PHYSICAL REVIEW E VOLUME 49, NUMBER 3 MARCH 1994

Structural stability and selection of propagating fronts in semilinear parabolic partial differential equations

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(Received 28 June 1993)

An alternative viewpoint for the selection problem in propagating front systems is presented. We propose that the selected solution can be identified through analysis of the structural stability of the solutions in question. A solution to a given equation is considered structurally stable if it suffers only an infinitesimal change when the equation (not the solution) is perturbed infinitesimally. Applying the structural stability condition, we identify selected solutions for semilinear parabolic partial differential equations, single-mode equations of the Fisher type, and multiple-mode equations that assume something of a generalized Fisher form. The structural stability condition is confirmed for the subclass of single-mode equations to which the Aronson-Weinberger theorem applies [in *Partial Differential Equations and Related Topics,* edited by J. A. Goldstein (Springer, Heidelberg, 1975)]. For other single-mode equations and multiple-mode equations, the structural stability condition is confirmed numerically. Equations possessing multiple physically realizable solutions are also studied, and several causes for such behavior are identified. We describe what we believe to be the fundamental feature distinguishing physically realizable and physically unrealizable solutions.

PACS number(s): 03.40.Kf, 68.10.Gw, 47.20.Ky

I. INTRODUCTION

Systems exhibiting front propagation phenomena have been the focus of considerable study in recent years. From the theoretical point of view, a very interesting aspect of this study has been the attempt to identify "selected" solutions to the equations modeling this behavior. Many of these equations possess multiple stable traveling-wave solutions, while the physical systems they describe exhibit reproducible behavior corresponding to only one of these solutions [1–9]. Presently, there exists no general method by which selected solutions of such equations can be identified. It is our goal to make progress toward the construction of such a method. In this paper, we identify a characterization of the selected solutions to a certain class of equations and isolate what we believe to be the fundamental feature distinguishing physically realizable and physically unrealizable solutions. With the insight afforded by this *selection principle,* we hope that further application of the theoretical framework used here will lead to methods by which selected solutions can be explicitly computed. Some attempts are already given in [10] along with a partial summary of this paper.

We address the question of selection by pursuing an approach which differs from those taken traditionally. We propose that for any propagating front system, the selected solution must be characterized by a sort of *structural stability,* whose definition will be given in Sec. III. Roughly speaking, this proposition amounts to the condition that a physically observable propagating front solution to a given equation must suffer only a small modification when the equation (not the solution) is altered slightly. This is our structural stability hypothesis.

In this paper, we study front propagation described by semilinear parabolic partial differential equations. The single-mode equations considered fall naturally into three distinct categories. We first consider equations of the Fisher form (2.1). The present theoretical understanding of such equations leads us to distinguish between two types, those to which the max-min principle of Aronson and Weinberger [12] apply (AW type) and the rest (non-AW type). In both cases, the structural stability hypothesis implies a characterization of the selected solutions as the slowest stable ones. For those equations of the AW type, this characterization is confirmed by the max-min principle. Numerical confirmation for a non-AW-type equation is given here. As a third type of equation, we consider a semilinear parabolic partial differential equation (PDE) which is not of the Fisher form and, based on numerical results, conclude that here again the selected solution is the slowest stable one. This result provides evidence that the arguments applied to the Fisher equation are valid for an even larger class of equations.

We next consider a somewhat restricted class of multiple-mode equations assuming something of a generalized Fisher form. Again, the structural stability hypothesis allows us to make a characterization of the selected solutions for such equations (see Sec. VI). In this case, although the characterization does not identify it as such, we believe that again the selected solution is the slowest stable one. A supporting numerical example is given. For one example of a less restricted class of multiple-mode equations, we obtain numerical results in

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TABLE I. The equations we study fall naturally into five distinct categories. The critical damping and minimum speed characterizations are equivalent for both types of Fisher equations as well as for the single non-Fisher equation studied. We conjecture that for the multiple-mode equations considered, these characterizations are again equivalent. For those equations possessing multiple physically realizable solutions representing invasion into a given unstable stationary solution, we believe that there exists a family of stable traveling wave solutions interpolating between this unstable stationary solution and a set of unique, isolated stable stationary solutions, and that each physically realizable solution represents the slowest member of such a family.

Equation type			Characterization	Confirmation
single-mode	Fisher	AW	critical damping	rigorous
		non-AW	=minimum speed	numeric
	non-Fisher			numeric
multiple-mode	single physical solution		critical damping (minimum speed[a])	numeric
	multiple physical solutions		critical damping (locally minimum speed[a])	

[a]Conjectured.

agreement with the hypothesis that the selected solution is structurally stable, and all others are structurally unstable. Finally, we study multiple-mode equations for which no unique selected solution exists. In this case, depending on the initial conditions, any one of multiple structurally stable solutions can be realized in a given experiment. We believe that each of these physically realizable solutions represents the slowest member of a distinct continuous family of solutions. Such a family consists of all stable solutions converging asymptotically behind their fronts to the same stable fixed point of the PDE. It is in this sense that the minimum speed hypothesis applies to such equations. A summary of the above discussion is given in Table I.

In the next section, we describe the nature of the selection problem for propagating front systems and briefly review some of the extant work relevant to the problem. The structural stability philosophy and its application to the Fisher equation are discussed in Sec. III. In Sec. IV, the selected solution for this equation is identified. We discuss the feature which distinguishes this solution from the other traveling-wave solutions and the relation between the structural stability and max-min arguments in Sec. V. The analysis applied to the Fisher equation is generalized in Sec. VI and applied to the multiple-mode equation discussed above. In Sec. VII a method to compute selected propagation speeds for certain N-mode systems is introduced. Numerical confirmation of the predictions of Secs. IV and VI for several equations is given in Sec. VIII. Systems which possess multiple physically realizable solutions are considered in Sec. IX. A short summary and concluding remarks are given in Sec. X.

II. SELECTION PROBLEM

Model equations describing front propagation are usually deterministic PDE's. Therefore, if we can characterize the class of initial conditions which are physically accessible (i.e., realizable under ordinary experimental conditions), solving a model equation as an initial value problem determines which propagating fronts are realizable. Thus when we consider the time evolution of the system, there is no need to consider selection rules for the final steady states. The selection problem arises when we seek to make the same determination without solving the initial value problem. Given the set of all propagating solutions to a model PDE, can we find a principle which allows us to identify the physically realizable solution(s) without explicitly solving the initial value problem? This is the selection problem we wish to address. The question is practically meaningful because it is often easier to obtain propagating front solutions than to solve initial value problems. In order to find such a principle, we need a way to characterize those wave fronts which are observable under natural experimental conditions.

The characterization of accessible propagating solutions has been the goal of considerable study for many years. Kolmogorov, Petrovskii, and Piskunov [11], Aronson and Weinberger [12], Hadeler and Rothe [13], and others have made important progress toward this goal through their study of the following semilinear parabolic PDE, often called the Fisher equation,

$$\frac{\partial\psi}{\partial t}=\frac{\partial^2\psi}{\partial x^2}+F(\psi) , \tag{2.1}$$

where F is a continuous function with $F(0)=F(1)=0$. If F furthermore satisfies the condition $F(\psi)>0$ for all $\psi\in(0,1)$, then there exists a stable traveling-wave solution interpolating between $\psi=1$ and $\psi=0$ with propagation speed c for each value of c greater than or equal to some minimum value c^*, where $c^*\geq c_0\equiv 2\sqrt{l_0}$, and $l_0\equiv F'(0)$. The positivity condition on F stated above together with the condition that $l_0>0$ will henceforth be called the AW condition.

For (2.1) with the AW condition in 1-space, Aronson and Weinberger proved that if the initial value $\psi(x,0)\in[0,1]$ vanishes beyond some finite x_0, then $\psi(x,t)$ converges to the traveling-wave solution of (2.1) with speed c^* in the following sense: for any $\beta\in(0,1)$, $\lim_{t\to\infty}X(t)/t=\lim_{t\to\infty}x(t)/t=c^*$, where $X(t)\equiv\max\{x:\psi(x,t)=\beta\}$ and $x(t)\equiv\min\{x>0:\psi(x,t)=\beta\}$. Since it is usually difficult to manipulate the initial conditions outside a compact set in any experiment (real or numeric), this result has been interpreted as proof that the slowest stable traveling-wave solution is the selected solution for (2.1). This result was extended to the Fisher equation with the AW condition in $d(>1)$-space in a conclusive paper by Weinberger [14] (actually, the paper contains much more general results). We can summarize Weinberger's result as the *minimum speed principle for semilinear parabolic PDE's:* the selected solution always corresponds to the minimum speed for which there exists stable front propagation.

Equations of the Fisher type (with the AW condition) fall into two distinct classes. For "pulled" equations, $c^*=c_0$, while for "pushed" equations, $c^*>c_0$ [15]. In the pulled case, the selected solution is marginally stable with respect to small, localized perturbations applied to its leading edge (see the Appendix for a discussion of stability). Essentially, this means that the propagation speed of the front is the same as that of these small perturbations.

Langer and others [16–18] studied several non-Fisher-type semilinear parabolic equations and found that these equations too apparently possess selected solutions which are marginally stable with respect to perturbations applied to their leading edges. We will refer to this type of marginal stability as "linear marginal stability," and the solution characterized by this type of stability will be termed "linearly marginally stable," following van Saarloos [18]. This led them to conjecture that, belying their nonlinear nature, there is a large class of equations with propagating front solutions for which speed and pattern selection are governed by linear order terms only, and that the selected solution to each such equation is linearly marginally stable. They were aware, however, that this conjecture is not generally true, noting that, for example, the selected solution of the equation

$$\frac{\partial\psi}{\partial t}=\frac{\partial^2\psi}{\partial x^2}+\psi+b\psi^2-\psi^3 \tag{2.2}$$

with sufficiently large $b>0$ has a speed larger than c_0.

The work of Langer and co-workers points out a common feature exhibited by the selected solutions for some front propagation problems and leads one naturally to ask in what systems such a feature can be found and how these systems differ from those in which it is lacking. Presently, however, there is no general theory which can answer these questions, and thus there is no general method to check the prediction provided by marginal stability theory. In certain cases, though, there are known methods by which the selected solution can be constructed explicitly. Using one such method, van Saarloos [18] was able to describe the breakdown of linear marginal stability theory and show how the nonlinear terms become important for a number of equations.

Some progress has been made toward the goal of extending the marginal stability idea to systems for which the linear analysis fails [18]. Here, it is hypothesized that in the nonlinear case there exists an "invasion mode" whose dynamics govern the evolution of the system toward its asymptotic solution. A physically appealing argument is then given that this mode corresponds to a particular speed, $c^\dagger$, in the sense that if the mode is allowed to grow, it will eventually travel at this speed. It is then conjectured that the necessary and sufficient condition for this growth to take place from a perturbation of some existing solution with speed c is that $c^\dagger>c$. Although it is not stated, it appears that implicit in this argument is the assumption that this so-called invasion mode is the fastest such mode which can be excited by a physically realizable perturbation. The conclusion then is that $c^\dagger$ is the marginally stable speed; all smaller speeds are unstable with respect to the invasion mode, and all larger speeds are stable with respect to this and all other modes which can be excited by physical perturbations.

Although the actual dynamics of these systems are probably too complicated to be described in terms of the propagation of such an invasion mode, this idea does provide an intuitive picture. It may be possible to develop the concept of nonlinear marginal stability further, but at this time, it has not been taken beyond the point of a feasible qualitative description.

In the following section, we discuss what we will term the *structural stability philosophy.* We show that application of this philosophy leads to a characterization of selected propagating front solutions.

III. STRUCTURAL STABILITY

Suppose one repeatedly performs the same experiment on a particular physical system and obtains a set of results. Since this system cannot be prepared or maintained identically for any two experiments, we expect these results to display a finite degree of variance. If the variance, however, is consistent within the bounds of experimental uncertainty, we will say that the system displays reproducibly observable phenomena.

The basic idea employed in this paper is that a good model of reproducibly observable phenomena must be *structurally stable,* i.e., the physical predictions provided by the *model* must be stable against modifications corresponding to perturbations of the *physical system* being modeled which alter its behavior only infinitesimally. (For convenience, we will refer to both small perturbations of the physical system and to the corresponding perturbations of the model as *physically small.)* Here notice that we are not perturbing a solution, but the structure of the system itself.

The idea of structural stability used here is close to the idea proposed by Andronov and Pontrjagin [19] for dynamical systems. In their formulation, the model equation is considered structurally stable if, crudely speaking, its dynamics are stable against any small change to its form. Of course, in the modeling of natural phenomena, we need not require the structural stability of the entire system, but may have only to require that of the solution corresponding to reproducibly observable phenomena. In this case we call the solution a *structurally stable solution.* Our conjecture is that reproducibly observable propagating fronts in physical systems correspond to only structurally stable solutions of (good) model equations. We wish to call this the structural stability hypothesis. While we believe that this hypothesis should hold for all equations meaningful in science, in this paper we apply it to the particular class of semilinear parabolic PDE's.

Our goal is to identify structurally stable propagating front solutions (i.e., structurally stable predictions) to certain semilinear parabolic PDE's by considering physically small perturbations of these PDE's and studying the resulting changes suffered by their solutions. Actually, since the propagation speed uniquely determines the solution in many cases, we have only to study the structural stability of the propagation speed. In what follows, we consider only perturbations to the function F in (2.1). Adding a second order time derivative to (2.1) is equivalent to modifying the diffusion constant for systems with traveling-wave (i.e., non-pattern-forming) solutions. A change in the diffusion constant can be absorbed into a scale change. Adding higher order spatial derivatives and time-space mixed derivatives with small coefficients may be interesting, but general experience with singular perturbation tells us that, excluding the possibility of bifurcation phenomena, such perturbations do not affect the very global nature of solutions. In particular, this statement is expected to hold for (2.1), and we will therefore not consider perturbations of this type.

With a wider application of our structural stability conjecture in mind, we should note here that bifurcation phenomena in general represents structurally unstable behavior. This phenomena, however, is nongeneric in the sense that it generally occurs at a very small subset of possible parameter values. Of course we have already implicitly excluded this subset from consideration, since a physical system described by an equation at a bifurcation point would not exhibit reproducibly observable phenomena.

Before continuing, let us note further that we will not consider perturbations which explicitly depend on time or space coordinates. In particular, we will not consider space-time inhomogeneous perturbations (noise), although it is natural to expect that structurally stable solutions are stable against such perturbations. This point is discussed in Sec. X.

It is easy to show that all propagating solutions of (2.1) are structurally stable against C^1-small perturbations δF of F (C^1 smallness requires not only δF but also $\delta F'$ to be small). If we wish for our method to isolate a unique physically realizable solution, then, we must consider more severe perturbations. Consider (2.1) as describing the propagation of fire along a fuse. F represents the net rate of heat production as a function of temperature, represented by ψ. $\psi=0$ corresponds to the ignition temperature, and $\psi=1$ corresponds to the maximum attainable temperature. The propagating front thus interpolates between the $\psi=0$ region before it, where fuel has not yet begun to burn, and the $\psi=1$ region behind it, where fuel is burning at a constant rate. (We can imagine that fuel is being added behind the front to sustain burning in this region.) It is reasonable that the observable properties of such a front would be insensitive to most small changes in the heat production rate, F. For example, choosing a single temperature between $\psi=0$ and $\psi=1$, changing the heat production rate in its neighborhood by a (vanishingly) small amount should alter the observable behavior only slightly. While the C^1 norm of such a perturbation can be made indefinitely large, its C^0 norm is small. It is thus natural to consider C^0-small (but C^1-large) continuous perturbations to F.

Many semilinear parabolic PDE's are not structurally stable with respect to C^0-small perturbations. The Fisher equation is not an exception. In fact, all of its propagating front solutions are structurally unstable against certain C^0-small perturbations of F. That is, we can always invent a small continuous perturbation to F which can cause an arbitrarily large change in the speed of a given propagating solution; for every number c greater than some positive minimum value, there exists a sequence of continuous perturbations converging to 0 in the C^0 norm, each element of which gives a unique traveling-wave solution, and whose corresponding sequence of speeds converges to c. If the Fisher equation is a good model of observable phenomena, this sensitivity to certain C^0-small perturbations must correspond to a similar sensitivity of the actual physical system being modeled.

Again let us consider the fire analogy. Altering F very near $\psi=0$ with a C^0-small continuous perturbation can result in extremely explosive behavior. For example, consider the case in which a small amount of explosive is sprinkled uniformly along the fuse. As a result, the rate at which heat production increases as a function of temperature, $dF/d\psi$, can be made arbitrarily large in the neighborhood of $\psi=0$, even in the limit of vanishing C^0 norm. In this case, the explosive low temperature behavior will travel very rapidly along the fuse, setting off the relatively sluggish higher temperature behavior behind it. The resulting temperature front will thus propagate with a very large speed and assume a very long, flat shape.

It is clear then that certain C^0-small perturbations are not physically small. The pathological behavior described above, however, results only from a perturbation which increases the quantity $\sup_{\psi\in(0,\eta]}[F(\psi)/\psi]$ appreciably for some $\eta>0$. We will call a C^0-small perturbation for which $\sup_{\psi>0}[\delta F(\psi)/\psi]$ is less than some small positive number (which goes to zero continuously as the C^0 norm of δF vanishes) a *p-small* perturbation [20]. Note that for a given η, the quantity $Q_\eta(F)\equiv\sup_{\psi\in(0,\eta]}[F(\psi)/\psi]$ is lower semicontinuous with respect to the C^0 norm of δF, but it is not upper semicontinuous. That is, $\inf_{\delta F}Q_\eta(F+\delta F)-Q_\eta(F)$ vanishes continuously with the C^0 norm of δF for any positive η, but $\sup_{\delta F}Q_\eta(F+\delta F)-Q_\eta(F)$ does not for some η. If the set of allowed δF is restricted by the p-smallness condition, however, upper semicontinuity of Q_η is insured. Thus p-small perturbations are those C^0-small perturbations for which the following continuity condition holds: $\lim_{\|\delta F\|\to 0}Q_\eta(\delta F)=Q_\eta(0)$ for any $\eta>0$. Our conjecture is that p-small perturbations are physically small for many phenomena described by semilinear parabolic PDE's. In any case, we confine ourselves to p-small perturbations only. [For convenience, we also require perturbations to satisfy $\delta F(0)=0$. See [21] for a related discussion.] The precise form of our structural stability hypothesis is that the physically realizable solutions of

(2.1) are those which are stable with respect to p-small structural perturbations.

IV. SELECTION OF SOLUTIONS

We will refer to semilinear parabolic PDE's possessing more than one stable traveling-wave solution for a given set of boundary conditions as "ambiguous" and those possessing a single such solution as "unambiguous." Our goal is to identify selected solutions to ambiguous PDE's. We wish to claim that given an ambiguous semilinear PDE, there exists a dense set (with respect to the C^0 norm of F) of unambiguous PDE's which can be obtained from the original by p-small perturbations [21]. The unique traveling-wave solutions to these unambiguous PDE's converge to a unique traveling-wave solution of the unperturbed ambiguous PDE when the C^0 distance from it is reduced to zero. This solution is thus, according to our hypothesis, the selected solution of the original PDE. The analysis of Secs. IV A and IV B applies to any ambiguous equation of the Fisher form (with boundary conditions specified) and allows us to identify the selected solution for each such equation.

A. Mechanical analogy

Assuming a traveling-wave solution $\psi(x,t)=\varphi(x-ct)\equiv\varphi(\xi)$ of (2.1), we can convert this PDE into the following ODE:

$$\frac{d^2\varphi}{d\xi^2}+c\frac{d\varphi}{d\xi}+F(\varphi)=0 . \tag{4.1}$$

As is well known, the most convenient way to understand (4.1) is in terms of the motion of a particle moving in a one-dimensional potential and subject to a frictional force with coefficient c. This analogy is made explicit through the identifications $\xi\to t$, $\varphi\to q$, $d\varphi/d\xi\to p\equiv\dot{q}$, and $\mathcal{F}=-F$. We then have

$$\dot{p}=-cp+\mathcal{F}(q) . \tag{4.2}$$

The force $\mathcal{F}(q)$ has an associated potential $V_0(q)$. In a typical ambiguous case, V_0 is monotonically increasing from $q=0$ to $q=1$. A traveling-wave solution to (2.1) interpolating between $\psi=0$ and $\psi=1$ corresponds to the trajectory of a particle falling from $q=1$ with zero initial kinetic energy and coming to rest at $q=0$. Cast in these terms, we can interpret c^* as the critical value of the frictional coefficient. If c is smaller than this value, the particle overshoots $q=0$. This case corresponds to a traveling-wave solution of (2.1) in which ψ assumes negative values. Clearly such a solution is unstable (in the conventional sense). If $\mathcal{F}\le 0$ for all $q\in[0,1]$, then for all $c\ge c^*$ the particle stops at $q=0$ without overshooting. (Without this condition on $\mathcal{F}$, there may be a supremum of such values of c.) Hence, for each propagation speed $c\ge c^*$ there exists a stable traveling-wave solution.

From this mechanical analogy, it is easy to see that for a semilinear parabolic PDE to be unambiguous, the potential associated with the corresponding ODE must not have a local isolated minimum at $q=0$. That is, either $q=0$ is a local maximum, or there is a finite neighborhood of $q=0$ where the potential is flat. In this case, there is a single value of c for which the particle comes to rest at $q=0$. We will call this value of c the *characteristic speed* of the potential. Furthermore, it is clear that if the potential has a local isolated minimum at $q=0$, it is inevitably ambiguous. Therefore, the presence of a local maximum or finite flat neighborhood at $q=0$ is a necessary and sufficient condition for the unperturbed form of (2.1) to be unambiguous. For the flat case the unambiguity is demonstrated in [22].

B. Selection

We now consider p-small perturbations of F which produce unambiguous PDE's from ambiguous ones. (As mentioned previously, the set of all such unambiguous PDE's is dense in the set of all PDE's which can be produced from the original by p-small perturbations.) To understand the effect of these perturbations on the propagating front behavior of the PDE's, we will consider their effect on particle trajectories in the mechanical analogy. In particular, we will identify the limiting value of the characteristic speed for such systems as the C^0 norm of these perturbations vanishes.

We will refer to the smallest value of c for which the particle comes to rest at $q=0$ without overshooting as the *critical speed.* In the case that $V_0(q)$ is unambiguous, the critical speed and the charcteristic speed are identical. We will now demonstrate that the critical speed depends continuously on any p-small perturbation of $\mathcal{F}(q)$. Thus c^* is the only structurally stable speed and, according to our hypothesis, is the selected speed for a propagating front solution of the original PDE.

Throughout this section we consider particle trajectories originating from the point of (global) maximum potential energy q_1 with zero kinetic energy, and leaving this point in the direction of decreasing q. (For all unperturbed systems, $q_1=1$.) To this point, c^* has been used to represent the critical speed of the unperturbed system. We now, however, wish to consider the critical speed of the system as a function of $\mathcal{F}$, and will therefore, in a natural generalization of notation, use $c^*(\mathcal{F})$ to represent this quantity. Also, in the following discussion, $p_q(\mathcal{F},c)$ will represent the velocity of the particle and $T_q(\mathcal{F},c)$ the kinetic energy, $p_q^2(\mathcal{F},c)/2$, for the system with force $\mathcal{F}$ and frictional coefficient c when the particle first arrives at q. For any given c and $\mathcal{F}$ we can define $q_0=\max\{0,\tilde{q}\}$, where $\tilde{q}$ is the first point (excluding q_1) at which the kinetic energy of the particle vanishes. Thus for any $q\in[q_0,q_1]$, $p_q(\mathcal{F},c)$ is well defined and nonpositive. Let us call $[q_0,q_1]$ the *negative velocity* interval.

For the proof given below, we need only consider negative velocity intervals. The following lemmas, which will be used in the proof, concern pairs of systems for which c and/or $\mathcal{F}$ differ. They hold for any point q belonging to the negative velocity intervals of *both* systems. With a slight change of notation, henceforth q_1 will represent the largest such point, and q_0 the smallest. Whenever there is no danger of ambiguity, we will use the abbreviation T_i in place of $T_q(\mathcal{F}_i,c)$.

(i) Consider two systems "1" and "2" whose frictional coefficients are identical, but whose forces $\mathcal{F}_1$ and $\mathcal{F}_2$ differ and for which $T_{q_1}(\mathcal{F}_1,c) \le T_{q_1}(\mathcal{F}_2,c)$. Then $\mathcal{F}_1 \ge \mathcal{F}_2$ implies $T_q(\mathcal{F}_1,c) \le T_q(\mathcal{F}_2,c)$. In particular, $T_0(\mathcal{F}_1,c) \le T_0(\mathcal{F}_2,c)$ if $0 \in [q_0,q_1]$ (i.e., if the particles reach $q=0$). □

This follows from $dT/dq = c\sqrt{2T} + \mathcal{F}$. Then

$$\frac{d(T_1-T_2)}{dq} = c(\sqrt{2T_1} - \sqrt{2T_2}) + \mathcal{F}_1 - \mathcal{F}_2 .$$

Suppose that $T_1 \le T_2$ is not maintained. Then there exists q at which $T_1 = T_2$ and $dT_1/dq < dT_2/dq$ (note that the particles are moving in the direction of decreasing q). However, since $\mathcal{F}_1 \ge \mathcal{F}_2$, this is impossible. Hence once $T_1 \le T_2$ is satisfied, T_1 can never again be larger than T_2. Then since initially, at $q=q_1$, $T_1 \le T_2$, the result follows.

(ii) Consider two systems governed by a single force $\mathcal{F}$ but frictional coefficients c_1 and c_2. Then if $T_{q_1}(\mathcal{F},c_1) \le T_{q_1}(\mathcal{F},c_2)$ and $c_1 \ge c_2$, $T_q(\mathcal{F},c_1) \le T_q(\mathcal{F},c_2)$. □

We have T_1, T_2 being defined as in (i),

$$\frac{d(T_1-T_2)}{dq} = (c_1\sqrt{2T_1} - c_2\sqrt{2T_2}) .$$

Then exactly the same argument as in (i) demonstrates (ii).

(iii) Consider two critically damped systems with forces $\mathcal{F}_1$ and $\mathcal{F}_2$, and suppose $T_{q_1}[\mathcal{F}_1,c^*(\mathcal{F}_1)] \le T_{q_1}[\mathcal{F}_2,c^*(\mathcal{F}_2)]$. Then $\mathcal{F}_1 \ge \mathcal{F}_2$ implies $c^*(\mathcal{F}_1) \le c^*(\mathcal{F}_2)$. □

This follows from (i) and (ii).

(iv) Suppose $q_0 < q_a < q_b < q_1$, $\sigma \ge 0$, and

$$\mathcal{F}_2 = \begin{cases} \mathcal{F}_1 + \sigma, & q \in (q_b, q_b + dq) \\ \mathcal{F}_1 - \sigma, & q \in (q_a, q_a + dq) \\ \mathcal{F}_1, & \text{otherwise} . \end{cases} \tag{4.3}$$

Then $T_{q_1}(\mathcal{F}_1,c) = T_{q_1}(\mathcal{F}_2,c)$ implies $T_q(\mathcal{F}_1,c) \le T_q(\mathcal{F}_2,c)$ for $q < q_a$ and any c (>0). □

For $q > q_a + dq$, $\mathcal{F}_2 \ge \mathcal{F}_1$, so that $T_q(\mathcal{F}_2,c) \le T_q(\mathcal{F}_1,c)$ due to (i). Hence the frictional force acting on the particle in system "1" dissipates no less energy than that acting on the particle in system "2" up to this point. After passing q_a, however, the total energies the systems have gained from their respective forces become identical. Thus immediately after passing q_a, $T_2 \ge T_1$, and since the systems are identical for all smaller q, the result is clear.

We split the demonstration of continuity of c^* into two parts.

(A) For any $\epsilon > 0$ there is $\delta > 0$ such that for any C^0 perturbation $\delta\mathcal{F}$ satisfying $c^*(\mathcal{F}) \ge c^*(\mathcal{F}+\delta\mathcal{F})$, $\|\delta\mathcal{F}\| < \delta$ implies $c^*(\mathcal{F}) - c^*(\mathcal{F}+\delta\mathcal{F}) < \epsilon$.

(B) For any $\epsilon > 0$ there is $\delta > 0$ such that for any C^0 perturbation $\delta\mathcal{F}$ satisfying $c^*(\mathcal{F}) \le c^*(\mathcal{F}+\delta\mathcal{F})$, $\|\delta\mathcal{F}\| < \delta$ and p smallness imply $c^*(\mathcal{F}+\delta\mathcal{F}) - c^*(\mathcal{F}) < \epsilon$.

Let us demonstrate (A) first. Due to (iii), even if the force $\mathcal{F}'$ is not continuous, the conditions $T'_{q_1} \le T_{q_1}$ and $\mathcal{F}' \ge \mathcal{F}$ imply that $c^*(\mathcal{F}') \le c^*(\mathcal{F})$. Hence for given δ there is a "maximal" perturbation $\delta\mathcal{F}_D$ with C^0-norm δ which decreases c^* by an amount more than any other perturbation of equal or smaller norm, namely

$$\delta\mathcal{F}_D = \begin{cases} \delta(>0) & q \in (0,1] \\ 0, & q = 0 . \end{cases} \tag{4.4}$$

In order for (i)–(iii) to hold for the unperturbed and $\delta\mathcal{F}_D$-perturbed systems, δ must be small enough that q_1 and q_0 as defined above exist. Such a δ can always be chosen. Now suppose $c = c' \equiv c^*(\mathcal{F}) - \epsilon$ in the original system with force $\mathcal{F}$. Then by definition, $T_0(\mathcal{F},c') > 0$. $T_0(\mathcal{F},c')$ depends on $\mathcal{F}$ continuously, and thus there is a positive δ such that $T_0(\mathcal{F}+\delta\mathcal{F}_D,c') \ge 0$. Hence (iii) implies $c^*(\mathcal{F}+\delta\mathcal{F}_D) \ge c^*(\mathcal{F}) - \epsilon$. But by construction, $c^*(\mathcal{F}+\delta\mathcal{F}) \ge c^*(\mathcal{F}+\delta\mathcal{F}_D)$. (A) has thus been demonstrated.

We next demonstrate (B). Given a perturbation $\delta\mathcal{F}$ with C^0-norm δ, we wish to consider its following modified forms:

$$\delta\mathcal{F}_1 = \begin{cases} -\delta & q \in [1-\beta, 1] \\ \delta\mathcal{F} , & \text{otherwise} \end{cases} \tag{4.5}$$

and

$$\delta\mathcal{F}_2 = \begin{cases} 0 , & q \in [1-\beta, 1] \\ \delta\mathcal{F} - 2\delta\beta , & q \in [\frac{1}{4}, \frac{3}{4}] \\ \delta\mathcal{F} , & \text{otherwise} , \end{cases} \tag{4.6}$$

where $0 < \beta < \frac{1}{4}$. Repeated use of (iv) shows $T_q(\mathcal{F}+\delta\mathcal{F}_1,c) \le T_q(\mathcal{F}+\delta\mathcal{F}_2,c)$ for all $q \in [0,\frac{1}{4}]$ and any c such that q is in the negative velocity interval of both systems. Hence $c^*(\mathcal{F}+\delta\mathcal{F}_1) \le c^*(\mathcal{F}+\delta\mathcal{F}_2)$. Then since $\delta\mathcal{F}_1 \le \delta\mathcal{F}$, $c^*(\mathcal{F}+\delta\mathcal{F}) \le c^*(\mathcal{F}+\delta\mathcal{F}_1)$ by (iii), and consequently $c^*(\mathcal{F}+\delta\mathcal{F}) \le c^*(\mathcal{F}+\delta\mathcal{F}_2)$.

Next, note that $|\delta\mathcal{F}_2| \le \delta(1+2\beta)$ for all $q \in [0,1]$. Also, since $\delta\mathcal{F}$ is p small, there exists a positive function $g(\delta)$ such that $|\delta\mathcal{F}_2(q)| \le g(\delta)q$ for all $q \in [0,1]$, where $\lim_{\delta\to 0} g(\delta) = 0$. Now we consider the speed of the particle in the original system with critical damping, $|p_q(\mathcal{F},c^*)|$, and that of the particle for the system with force $\mathcal{F}+\delta\mathcal{F}_2$ and frictional coefficient $c^*+\epsilon$, $|p_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)|$. We will show that for any given ϵ, there exists a sufficiently small δ such that $|p_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)| \le |p_q(\mathcal{F},c^*)|$ for all $q \in (0,1]$.

We begin by noting that since $\epsilon > 0$, $|p_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)| \le |p_q(\mathcal{F},c^*)|$ for all $q \in [1-\beta,1]$. In what follows, we will show that even with infinitesimal positive ϵ, δ can be chosen such that $|p_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)| \le |p_q(\mathcal{F},c^*)|$ everywhere. This will be done by showing that if $|p_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)| = |p_q(\mathcal{F},c^*)|$ at an arbitrary point q, then the increased amount of energy lost in the dq neighborhood of this point due to the change $c^* \to c^*+\epsilon$ is greater than the increased amount of energy gained due to the change $\mathcal{F} \to \mathcal{F}+\delta\mathcal{F}_2$.

By considering the linearized equation near $q=0$ of the critically damped particle in the unperturbed system, we find that $|p_q(\mathcal{F},c^*)| \rightarrow [c^*/2+\sqrt{(c^*/2)^2-l_0}]q$ as $q\rightarrow 0$. Thus for sufficiently small δ, $|p_q(\mathcal{F},c^*)|>\delta^{1/2}$ for any $q\in[\delta^{1/3},1-\beta]$ [call this condition (a) for δ]. Now assume $|p_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)|=|p_q(\mathcal{F},c^*)|$ for some $\hat{q}\in[\delta^{1/3},1-\beta]$. In the neighborhood surrounding $\hat{q}$ of size dq, the change of the energy gain due to $\delta\mathcal{F}_2$ is not more than $\frac{3}{2}\delta dq$, while the change in energy loss due to the change $c^*\rightarrow c^*+\epsilon$ is not less than $\epsilon|p_q(\mathcal{F},c^*)|dq$. Hence if $\frac{3}{2}\delta<\epsilon\delta^{1/2}$ [condition (b) for δ], for values of q infinitesimally smaller than $\hat{q}$, $|p_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)| < |p_q(\mathcal{F},c^*)|$. Thus for $q\geq\delta^{1/3}$, $|p_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)| \leq |p_q(\mathcal{F},c^*)|$.

We now consider $q<\delta^{1/3}$. The linearized estimate for $|p_q(\mathcal{F},c^*)|$ can then be made arbitrarily accurate by choosing δ as small as necessary. In particular, it can be chosen small enough that for all $q<\delta^{1/3}$, $|p_q(\mathcal{F},c^*)|\geq c^*q/4$ [condition (c) for δ]. Now, assume $|p_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)|=|p_q(\mathcal{F},c^*)|$ for some $\hat{q}\in(0,\delta^{1/3}]$. Then the change of the energy gain in the dq neighborhood of this point due to the perturbation of $\mathcal{F}$ is no greater than $g(\delta)\hat{q}\,dq$, while the change of energy loss due to the change of c is no less than $\epsilon c^*\hat{q}\,dq/4$. Thus if $g(\delta)<\epsilon c^*/4$ [condition (d) for δ], then $|p_q(\mathcal{F},c^*)|>|p_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)|$ for values of q infinitesimally smaller than $\hat{q}$. δ can always be chosen small enough that conditions (a)–(d) are satisfied. Hence $T_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)\leq T_q(\mathcal{F},c^*)$ for all $q\in(0,1]$. Thus either $T_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)=0$ at some $q>0$, or $\lim_{q\rightarrow 0} T_q(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)=0$. In the latter case, due to the continuity of the kinetic energy, $T_0(\mathcal{F}+\delta\mathcal{F}_2,c^*+\epsilon)=0$. In either case we conclude that $c^*(\mathcal{F}+\delta\mathcal{F}_2)\leq c^*+\epsilon$. We have thus demonstrated (B).

An intuitive understanding of the continuity theorem we have just proven can be obtained by considering the following. Since the origin of the unperturbed particle system is a local minimum of the potential, there exist multiple trajectories corresponding to stable traveling-wave solutions of the PDE. If we place a small bump peaked at the origin, however, only one of these survives. If this bump represents the only perturbation of the system (that is, if the perturbation serves only to slow the particles), then the critically damped trajectory and all overdamped trajectories are destroyed (kept from reaching the origin). But in the limit that the size of this bump goes to zero, the value of c corresponding to the unique trajectory terminating at the origin (without overshooting) converges to c^*, the minimum speed of the original PDE. In this sense, the critical trajectory is restored. In fact, we have shown that even in the case that the origin remains a local minimum, the speed of the slowest stable solution always converges to c^* as the C^0 norm of the perturbation vanishes. The smallest speed solution is therefore structurally stable. For discussion of the reasoning which motivated the above continuity theorem, see [23].

We have shown that for the Fisher equation, the structural stability hypothesis implies the minimum speed characterization. With the AW condition, this characterization is rigorously established. The structural stability hypothesis is thus confirmed for equations of the Fisher form satisfying the AW condition.

C. Comment: Structural stability of ODE's

Since a traveling-wave solution of (2.1) corresponds to a saddle connection in the $(\varphi,d\varphi/d\xi)$-flow field corresponding to the ODE (4.1), there may exist the misconception that such a solution cannot be structurally stable. To avoid such a misconception, we summarize here the result of Sec. IV B in terms of (4.1) and its flow field.

For the unperturbed ODE, there is a saddle connection terminating at (0,0) corresponding to a stable traveling-wave solution of (2.1) for each value of $c\geq c^*$. If we consider some $c>c^*$, perturbing (4.1) with a p-small $\delta F(\varphi,\delta)$ whose application to (2.1) results in an unambiguous PDE, then for all sufficiently small δ, the saddle connection is destroyed. In addition, even in the $\delta\rightarrow 0$ limit, this connection can be restored only if c is altered by a finite amount. Next, we consider the case $c=c^*$. Applying the same $\delta F(\varphi,\delta)$, the saddle connection can be destroyed, but in this case, in the $\delta\rightarrow 0$ limit it can always be restored by altering c only infinitesimally. Thus considering the totality of flow fields in $(\varphi,d\varphi/d\xi,c)$ space, the only structurally stable saddle connection corresponding to a stable traveling-wave solution of (2.1) is that with $c=c^*$.

V. PHYSICAL IMPLICATIONS

By imposing the condition of structural stability, we have been able to single out a particular traveling-wave solution to (2.1) as being the only meaningful one physically. For the more restricted case in which the AW condition is imposed, this solution is the same as that found by Aronson and Weinberger [12]. This fact is reassuring, but at the same time it is somewhat mysterious. The physical arguments used in this paper and those used by Aronson and Weinberger are completely different. In this section we consider the time evolution of the PDE (2.1) and discuss in this context the nature of physically realizable and unrealizable traveling waves. From this discussion, we are able to identify the fundamental feature distinguishing such solutions, and gain an understanding of the physical implications of structural stability in the present problem. As a result, it becomes intuitively clear why the slowest solution to (2.1) is structurally stable. Although we restrict the present discussion to AW-type equations, we believe the conclusions to hold for a more general class of semilinear parabolic PDE's.

Consider an equation of the form (2.1) satisfying the AW condition, and suppose its selected solution $\phi^*(x,t)$ has speed c^*. (Throughout this section, the variable ϕ will be used to represent asymptotic solutions, i.e., traveling waves, while ψ will be used to represent solutions of the PDE evolving from initial conditions set at some finite time.) Viewed from the frame moving at speed c^*, $\phi^*(x,t)$ approaches zero asymptotically as $\sim e^{-k^*x}$ for some positive k^*, as can be inferred from the linearized form of the original equation around $\psi=0$. (Without loss of generality, we will consider only fronts which propagate in the $+x$ direction.) We now argue that the fundamental feature which distinguishes c^* from all $c>c^*$ is that it is *the* propagation speed which is determined by the bulk, that is, the region behind the front's leading edge. In the pushed case this is not surprising, but the point is that this distinction holds even in the pulled case.

For a given solution $\psi(x,t)$, we will say that at time t_0 the leading edge decays as $\sim e^{-kx}$ if $\limsup_{x\to\infty}[\ln\psi(x,t_0)/x]=-k[k\in(0,\infty)]$. When this is $-\infty$, or when $\psi=0$ beyond a finite domain, we will say that the leading edge is trivial. We define the leading edge of $\psi(x,t)$ to be the region displaying this asymptotic behavior. It should be noted that in general, this leading edge does not contain the entire "linear region," i.e., the region in which ψ is small enough that it evolves according to the linearization of (2.1).

Suppose that we have a set of initial conditions $\psi(x,0)$ $(0\le\psi\le 1)$ which decay as $\sim e^{-kx}$ in the $+x$ direction. If $k>k^*$, the analysis of Aronson and Weinberger [12] can be used to show that the speed of $\psi(x,t)$ converges to c^* as $t\to\infty$ in the sense described in Sec. II. According to the comparison theorem [24], convergence in the present case must be at least as rapid as in the case that $\psi(x,0)$ is confined to a finite support. Despite this fact, the leading edge of $\psi(x,t)$ retains the form $\sim e^{-kx}$ for all time. This leading edge will propagate with speed $c=k+l_0/k$, where $l_0=F'(\psi)|_{\psi=0}$, as seen from the linearized equation around $\psi=0$. As is well known [18], $k^*\ge l_0^{1/2}$, and thus $c>c^*=k^*+l_0/k^*$. This holds for any value of $k>k^*$, and thus in each such case, the form of the leading edge of $\psi(x,t)$ is well defined, and the convergence of its speed to c^* is independent of this form.

Next, consider $\psi(x,0)$ as described above, but with k satisfying $k_0<k<k^*$. Here, $k_0=\sup(Q)$, where Q is the set of all wave numbers characterizing the asymptotic region of those stable traveling-wave solutions with $c>c^*$. (Only in the pulled case is $k_0=k^*$ [18]). In this case, a straightforward extension of the arguments used by Aronson and Weinberger shows that the propagation speed of $\psi(x,t)$ again converges to c^* [25]. This is in spite of the fact that the solution's leading edge retains the form $\sim e^{-kx}$ for all time. Thus again, for initial conditions of the type in question, convergence of the propagation speed to c^* is independent of the form of the solution's leading edge.

Finally, in the case that $\psi(x,0)$ is defined on a compact support, the form of the leading edge is trivial for all time. These three cases demonstrate that the speed c^* is characteristic of those solutions whose asymptotic nature is independent of the form of their leading edge. In this sense, c^* is selected by the bulk.

We next consider initial conditions $\psi(x,0)$ decaying asymptotically as $\sim e^{-kx}$, where $k<k_0$. Here again, we can apply a straightforward extension of the arguments used by Aronson and Weinberger to show that the speed of $\psi(x,t)$ converges to $c=k+l_0/k>c^*$ [25] (see also [26]). Conversely, to produce a front with speed c greater than c^*, initial conditions decaying as $\sim e^{-kx}$, where $k=\frac{1}{2}(c-\sqrt{c^2-4l_0})$, are required. Thus propagation speeds greater than c^* are characteristic of solutions whose asymptotic nature is determined by the form of their leading edges.

We would like now to contrast the present point of view with what seems to be a commonly held misconception concerning the role played by the tip in the evolution of a solution of (2.1). Considering the pulled case, the selected speed of (2.1) can be calculated correctly by considering only the asymptotic (in x) region of the solutions and the linearization of (2.1) about $\psi=0$. Such a calculation yields both the speed and the asymptotic form, i.e., the leading edge, of the selected solution. In this case, since the correct speed can be obtained by considering only the asymptotic region, it is perhaps natural to conclude that it is this region which controls the evolution of the solution $\psi(x,t)$. That is, the leading edge behaves in whatever manner it "desires," independent of the bulk region behind it which is simply pulled along at the speed chosen by the leading edge. We will now show why this conclusion is incorrect.

It is first important to note that the asymptotic form of solution obtained from the linear calculation referred to in the previous paragraph is that of the steady state solution. When we consider an actual solution of the PDE evolving toward this selected steady state, the form of its leading edge and that of the steady state solution are, in general, unrelated. As discussed above, the leading edge form of a solution to (2.1) is for all time determined by the initial conditions. In fact, the quantity $\sigma=\limsup_{x\to\infty}[\ln\psi(x,t_0)/x]$ is independent of time. This statement holds for both the case in which the speed c^* is realized asymptotically and that for which some larger speed is realized. If $\sigma\in[-\infty,-k^*]$, the asymptotic (in t) form approached by $\psi(x,t)$ is $\phi^*(x,t)$. Convergence toward this steady state proceeds from the bulk toward the tip (except, of course, in the case where $\sigma=-k^*$). The $\exp(-k^*x)$ form characterizing the selected solution begins to develop in the bulk, and the steady state is approached as this form pushes out toward the asymptotic region. In the process, the maximal amplitude of the asymptotic region gradually decreases toward zero. Despite this bulk invasion, of course, the leading edge of $\psi(x,t)$ whose form and speed c $(c>c^*)$ are set by the initial conditions persists for all time. Thus although it is true that this leading edge behaves in whatever manner it desires, and the speed at which it propagates is independent of the bulk behavior, the bulk is not pulled along at this speed, and in fact convergence to $\phi^*(x,t)$ is unconstrained by the leading edge.

In the case that $\sigma>-k^*$, the speed approached asymptotically is greater than c^*. Here again, σ is time independent, but in this case, the asymptotic (in x) form of the solution to the PDE and that of the steady state to which it is converging are identical. For such values of σ, convergence toward the steady state proceeds from the tip toward the bulk, and the bulk is truly pulled along at the speed determined by the leading edge.

We conclude that the speed c^* is realized when the exponential decay characteristic of the steady state is allowed to originate in the bulk and spread forward toward the tip. When this behavior is prevented by the initial conditions, $c>c^*$ behavior results. In the former case, the form approached by $\psi(x,t)$ as $t\to\infty$ is independent of its leading edge. In the latter case, this form is dictated by it.

We are now afforded a qualitative understanding of the meaning of structural stability as it applies to (2.1). A traveling-wave solution to (2.1) (with the AW condition) is structurally stable if and only if its speed is determined by the front's bulk rather than its leading edge. This conclusion is intuitively very appealing, and in fact we believe that it holds not only for (2.1), but indeed quite generally. Since the bulk behavior displayed by any given model equation corresponds to physical phenomena which, in principle, should be measurable, it must be the case that the bulk behavior of a structurally stable solution is altered only slightly by a small perturbation. On the other hand, tip behavior does not correspond to any reproducible, physically measurable quantity, and therefore structural stability of a solution does not require the form of the tip to be stable against small perturbation. In fact, it is very reasonable that in general the tips of propagating front solutions are easily destroyed (i.e., altered drastically) by certain small perturbations. For this reason, any solution whose physically observable quantities (e.g., propagation speed) depend on the nature of the tip should be structurally unstable. Those solutions whose physically observable quantities depend on the bulk, however, should be structurally stable. Based on this reasoning alone, it is also an intuitively appealing conclusion that only these structurally stable solutions represent physically reproducible behavior. Once again let us consider the fire-fuse analogy to illustrate this point. We imagine the fuse to be covered with a very thin film of water which quickly evaporates when heated to a temperature slightly above $\psi=0$. As a result, the flammability of the fuse is slightly decreased, but only very close to its ignition temperature. For the hypothetical large speed propagating flames, though, this small perturbation would be fatal. That is, even if it were somehow possible to give the unperturbed system the unphysical initial conditions required to realize a solution with $c>c^*$, it would not be possible to do so for the perturbed system. This perturbation singles out—even in the hypothetical situation in which any initial conditions could be realized—that speed which can be realized by the unperturbed system given physical initial conditions. Such a perturbation can therefore only slightly alter the realizable propagation of fire along the fuse.

For structurally unstable solutions, the form of the asymptotic decay of the leading edge determines the speed, which in turn determines the interface profile. For this reason, if we apply to (4.2) a perturbation such as

$$\delta V(q,a)=\begin{cases} a-dq^2\ , & |q|<(a/d)^{1/2} \\ 0\ , & |q|\geq(a/d)^{1/2}\ , \end{cases} \tag{5.1}$$

with $d>d^2V_0/dq^2|_{q=0}$, then in the $a\to 0$ limit, all solutions are destroyed except that with $c=c^*$. In this limit, the perturbation of (2.1) corresponding to the bump at $q=0$, (5.1), affects only the leading edge of any given propagating front. Only that single propagation speed which is selected by the bulk, and not the leading edge, can survive such a perturbation. It is clear that such a solution represents the slowest stable propagating front since any slower front must be unstable with respect to the bulk invasion.

The above discussion has lead us to the following conclusion: for semilinear parabolic equations, any set of physical initial conditions produces a solution which converges to a traveling wave whose nature is independent of the form of this solution's leading edge. Because of this independence, this traveling wave represents a structurally stable solution.

With the present understanding, it becomes intuitively clear why the propagation speed c^* is easily observed when (2.1) is numerically integrated. If this speed were determined by the tip of the front, discretization, no matter how fine, would represent a drastic perturbation, and we would expect observable propagation speeds to differ from c^* appreciably. Since c^* is determined by the bulk, however, discretization represents only a very mild perturbation, and in fact, independently of the details of the particular numerical scheme used, propagation speeds arbitrarily close to c^* can be observed.

VI. MULTIPLE-MODE SYSTEMS

We saw above that the selected solution to any PDE of the form (2.1) corresponds to the critically damped particle trajectory described by (4.2). Cast in terms of classical mechanics, the special nature of the propagation speed c^* becomes apparent. By applying a slight perturbation to $V_0(q)$, all trajectories can be "destroyed" (i.e., kept from reaching the origin) except that one corresponding to a particular value of c slightly different from c^*. There is nothing in this statement, however, that applies uniquely to particles in 1-space. It thus seems quite likely that the structural stability arguments could be applied to systems of equations reminiscent of (2.1), but describing the invasion of several coupled modes, in much the same way they were applied above.

In analogy to the previous section, we conjecture that for multiple-mode systems, p-small perturbations are physically small [27]. We then note that for any sequence of perturbations converting an ambiguous N-mode PDE into a sequence of unambiguous PDE's, there corresponds a unique sequence of trajectories representing stable traveling-wave solutions. Our structural stability conjecture is that if these perturbations are p small and their C^0 norm converges to 0, the associated sequence of propagation speeds converges to the selected speed of the original, unperturbed PDE.

Before proceeding, we note that we have not yet been able to obtain a characterization of the structurally stable solutions for systems possessing linear order coupling terms. We are thus forced to consider only linearly decoupled equations. In this case, we can again establish the critical damping characterization of structurally stable propagating solutions. Equations exhibiting linear order coupling terms will be discussed later.

We will study N-mode systems whose ith mode behaves according to

$$\frac{\partial\psi_i}{\partial t}=D_i\nabla^2\psi_i+F_i(\psi_1,\psi_2,\ldots,\psi_N)\ . \tag{6.1}$$

Without loss of generality, we assume that $\psi_0 \equiv (0,0,\ldots,0)$ and $\psi_1 \equiv (1,1,\ldots,1)$ represent unstable and stable stationary solutions of (6.1), respectively. We will study traveling-wave solutions to (6.1) interpolating between ψ_0 and ψ_1. We assume that $F_i = b_i\psi_i +$ (higher order terms). Thus in a sufficiently small neighborhood of the origin, these equations are virtually decoupled. We wish to study systems in which ψ_0 is linearly unstable with respect to perturbation by any mode. Thus we assume $b_i > 0$ for all i. We consider the case in which there is a unique direction of propagation along which all modes propagate with a single speed. In addition, we assume that the spatial variation of each mode is confined along the direction of propagation. (6.1) can then be written as an ODE describing the motion of a particle in N space if we make identifications analogous to those which led to (4.2) for the $N=1$ case. For the ith mode,

$$D_i \dot{p}_i = -cp_i + \mathcal{F}_i(\mathbf{q}) \ . \tag{6.2}$$

This ODE becomes effectively decoupled into N independent second order equations in some neighborhood D of the origin.

For any solution to (6.1) of interest, the corresponding particle stops at the origin. Therefore its motion is eventually confined to D. Although we cannot claim that all modes are positive in this domain, stability of the solution in the ordinary sense requires the existence of a small neighborhood of the origin in which no mode changes sign.

In N space, the particle trajectories again depend on the frictional coefficient c, but now, they also depend on the initial direction $\mathbf{n}$ which the particle takes as it leaves the starting point, $\mathbf{q}_1 = (1,1,\ldots,1)$. (In those cases in which the force acting on the particle can be derived from a potential, this point can be interpreted as the top of an N-dimensional hill.) Because of this, the idea of critical damping requires some special attention.

For the trajectory corresponding to $\mathbf{n}$ and c, denoted by $(\mathbf{n},c)$, we define $\tau_i(\mathbf{n},c)$ to be the largest time at which the coordinate q_i changes sign (arbitrarily defining $t=0$ to be that time at which $q_1 = \frac{1}{2}$). If q_i never changes sign, we define $\tau_i(\mathbf{n},c)=0$. If q_i changes sign at indefinitely large times, $\tau_i(\mathbf{n},c)=\infty$. As noted above, such a trajectory corresponds to an unstable solution of the PDE. We will call the trajectory $(\mathbf{n}_0,c_0)$ critically damped if $\tau_i(\mathbf{n}_0,c_0)$ is finite (or zero) for all i, and if there is a sequence S of trajectories which converges to $(\mathbf{n}_0,c_0)$ such that $S\text{-}\lim_{(\mathbf{n},c)\to(\mathbf{n}_0,c_0)} \tau_i(\mathbf{n},c) = \infty$ for all i, where $S\text{-}\lim_{(\mathbf{n},c)\to(\mathbf{n}_0,c_0)}$ denotes that the limit is taken by approaching $(\mathbf{n}_0,c_0)$ along S.

For $(\mathbf{n},c)\in S$, let us designate by $T_i(\mathbf{n},c)$ the kinetic energy associated with the ith coordinate at the last time it changes sign. If no such time exists (i.e., if this coordinate changes sign indefinitely), define $T_i(\mathbf{n},c)=0$. Then note that $S\text{-}\lim_{(\mathbf{n},c)\to(\mathbf{n}_0,c_0)} T_i(\mathbf{n},c)=0$. Also note that if no coordinate of a given trajectory $(\mathbf{n}_0,c_0)$ ever changes sign, and if there exists a sequence of trajectories S converging to $(\mathbf{n}_0,c_0)$ such that each mode of each trajectory changes sign at least once, then $(\mathbf{n}_0,c_0)$ must be critically damped. This simply follows from the fact that the kinetic energy associated with q_i [measured at any given $(\mathbf{q},t)\in(\mathbf{n},c)$ which changes continuously as a function of $(\mathbf{n},c)$] must be a continuous function of $(\mathbf{n},c)$. Thus the sequence of "overshooting" trajectories S must satisfy $S\text{-}\lim_{(\mathbf{n},c)\to(\mathbf{n}_0,c_0)} \tau_i(\mathbf{n},c) = \infty$ for all i.

We will now show that the structural stability hypothesis implies that a physically realizable solution to any equation of the form (6.1) must correspond to a critically damped trajectory. We demonstrate this by showing that a particular type of small perturbation placed near the origin is able to destroy all trajectories satisfying $\tau_i < \infty$ for some i. Essentially, we use the fact that near the origin, the N-dimensional trajectory becomes decoupled into N one-dimensional trajectories, and if $\tau_i < \infty$, near $q_i = 0$ the ith such trajectory is analogous to a $c \geq c^*$ trajectory of the Fisher equation. For this reason, trajectories for which $\tau_i < \infty$ for all i but which are not critically damped will be referred to as overdamped.

We consider traveling-wave solutions of (6.1) interpolating between ψ_0 and ψ_1. A trajectory starting at $\mathbf{q}_1$ and corresponding to a physically realizable solution of this type must therefore satisfy $\lim_{t\to\infty} q_i(t) = 0$ for all i. Since any solution of (6.1) for which $\tau_i = \infty$ for some i is (linearly) unstable, for any physically realizable solution, all τ_i must be finite. In order to show the desired result, we will consider certain p-small perturbed forms of (6.1) and show that for such equations, in the limit of vanishing C^0 norm, the sequence of τ_i corresponding to any sequence of surviving trajectories (i.e., trajectories which are able to come to rest at the origin) diverges for all i. Then by the definition of critical damping, and according to our structural stability hypothesis, we are able to conclude that physically realizable solutions to (6.1) must correspond to critically damped trajectories.

We limit our study to those systems for which any trajectory corresponding to a physically realizable solution of the PDE does not converge to the origin at any finite time. The pathological case in which $\lim_{t\to t_0} q_i(t) = 0$ for all i and some finite t_0 is ignored. Clearly, for linearly decoupled systems, such a case would require a "finely tuned" PDE which, upon slight perturbation would no longer display this pathological behavior. In this sense, such a situation is not generic.

Consider a perturbation of the particle equation of motion given by

$$\mathcal{F}_i(\mathbf{q}) \to \mathcal{F}_i(\mathbf{q}) - \frac{\partial}{\partial q_i} \delta V(\mathbf{q}) \ . \tag{6.3}$$

We choose δV to take the form of a small bump around the origin. More precisely, $\delta V(\mathbf{q})$ is continuously differentiable, and $\delta\mathcal{F}_i(\mathbf{q}) \equiv (\partial/\partial q_i)\delta V(\mathbf{q})$ satisfies the following properties for all i: $\delta\mathcal{F}_i(\mathbf{q})=0$ for $|\mathbf{q}| \geq \delta$, $\max|\delta\mathcal{F}_i(\mathbf{q})| \leq \delta$, and $\text{sgn}[\delta\mathcal{F}_i(q_i)] = \text{sgn}(q_i)$ at all points, where $\text{sgn}(x) = +1$ for $x \geq 0$, and -1 otherwise. Note that in the $\delta \to 0$ limit, the force to which the particle is subject for $|\mathbf{q}| < \delta$ is derivable from the potential

$$V(\mathbf{q})=\frac{1}{2}\sum_i b_i q_i^2+\delta V(\mathbf{q}) \ . \tag{6.4}$$

We choose $\delta V(\mathbf{q})$ so that $(\partial^2/\partial q_i^2)V(\mathbf{q})|_{q_i=0}<0$ for all $|\mathbf{q}|<\delta$ and all i, and so that very near the origin, $V(\mathbf{q})$ reduces to $V(\mathbf{q})=-\frac{1}{2}\sum_i a_i q_i^2$, where $a_i>0$ for all i. Such a perturbation can be constructed for any equation considered in this section. We give an illustration here for the case $N=2$. Define $\delta V=\delta V_1+\delta V_2$, where

$$\delta V_1(\mathbf{q},\delta)=\begin{cases}\delta^2 a_1\left[1+\cos\left[\frac{\pi}{\delta}q_2\right]\right][1+\cos(k_1 q_1)] \ , & |q_2|<\delta, \ |q_1|<\pi/k_1 \\ 0 \ \text{elsewhere} \ , & \end{cases}$$
$$k_1=\alpha_1\pi\left\{\delta\left[1+\cos\left[\frac{\pi}{\delta}q_2\right]\right]^{1/2}\right\}^{-1} \ , \tag{6.5}$$

and δV_2 is obtained by exchanging 1 and 2 in (6.5). Then, for example, setting $a_i=\min\{1/200b_i, b_i/200\}$ and $\alpha_i=\max\{10b_i, 10/b_i\}$, all of the above conditions are satisfied.

Consider some unperturbed trajectory $(\mathbf{n},c)$ for which $\tau_i(\mathbf{n},c)=\tau_i<\infty$. For any such trajectory, we can consider a time large enough that, to arbitrary precision, for all later times the trajectory is decoupled into N independent one-dimensional trajectories. Then, considering the system perturbed by a δV such as that constructed in the preceding paragraph, we can choose δ small enough that the particle does not encounter the perturbation until arbitrarily large time. We can thus think of the trajectory as first decoupling, and at a much later time suffering a perturbation. In the $\delta\to 0$ limit, the perturbation is confined to an infinitesimally small neighborhood of the origin, and for all trajectories of interest, the particle reaches this neighborhood with infinitesimally small kinetic energy. Such a trajectory is therefore unable to return at later times to the unperturbed coupled region. In the same limit, the time t_0 at which the particle first reaches the perturbed region satisfies $t_0>\tau_i$ for any finite τ_i. Near the origin then, and for $|\mathbf{q}|\geq\delta$, the particle in question is completely analogous to that studied in Sec. IV A and IV B with $c\geq c^*$. For $|\mathbf{q}|<\delta$, $\delta\mathcal{F}_i$ can only repel q_i from 0. Thus in the perturbed case as the unperturbed case, q_i cannot reach 0 at finite time.

Now, assume that this perturbed trajectory converges to the origin in the $t\to\infty$ limit. Then in this limit, the system again decouples into N independent trajectories. Considering q_i ($\neq 0$), we have $D_i\dot{p}_i=-cp_i+a_i q_i$. A simple analysis shows that $\lim_{t\to\infty}[q_i(t)]=0$ requires $|p_i(q_i)|>|p_i^0(q_i)|$ for all such q_i, where $|p_i^0|$ is the particle speed in the unperturbed case. However, since $\mathrm{sgn}[\delta\mathcal{F}_i(\mathbf{q})]=\mathrm{sgn}(q_i)$, in fact $|p_i(q_i)|\leq|p_i^0(q_i)|$ at all q_i along the trajectory. Thus $\lim_{t\to\infty}q_i(t)\neq 0$, and this trajectory is destroyed by the perturbation. We thus conclude that any unperturbed trajectory of interest for which $\tau_i(\mathbf{n},c)=\tau_i<\infty$ for some i can be destroyed by the type of perturbation in question for arbitrarily small $\delta>0$. The selected solution must therefore correspond to a critically damped trajectory.

While critical damping is a necessary condition for structural stability, in the multiple-mode case we cannot claim that it is sufficient. Because they are not confined to a single spatial dimension, we cannot exclude the possibility that there exist trajectories for multiple-mode systems which, although critically damped, are not able to survive certain p-small perturbations. (That is, for such perturbations, no trajectory in the neighborhood of this critically damped trajectory terminates at the origin.) Also, structural stability is not in general a sufficient condition for physical realizability in multiple-mode systems. In contrast to single-mode systems, structurally stable solutions of certain PDE's can be unstable in the conventional sense. In any case, if for a given PDE there exist no solutions stable in both senses, then this equation does not provide a good model of reproducible physical phenomena.

We should also note that the arguments given here do not necessarily support the minimum speed conjecture in the multiple-mode case. In particular, there may exist a direction $\mathbf{n}$ which possesses no critically damped trajectory, but for which there exist overdamped trajectories corresponding to values of c smaller than that of any critically damped trajectory. If such a case exists, however, it is likely that the solutions of the PDE corresponding to such slow overdamped trajectories would be unstable with respect to a faster critically damped solution. Thus we believe that even in the multiple-mode case, the minimum speed characterization remains valid.

In summary, for an N-mode linearly decoupled semilinear parabolic PDE providing a good model of a physical system which exhibits repeatable propagating front phenomena, the structural stability hypothesis implies that the selected solution must be critically damped. In addition, because for the multiple-mode equations studied here, as the single-mode equations studied earlier, only the physically realizable solutions are able to survive perturbation by a tiny "bump" at the origin, we believe that only these solutions are "bulk driven." For this reason, it is quite likely that any front propagating at a speed slower than the physically realizable one would be unstable with respect to the bulk invasion. Thus we conjecture that here again, the critical damping and minimum speed characterizations are equivalent.

In Sec. IX, we will consider PDE's for which multiple physically realizable solutions exist. For such equations there are natural families of solutions, each corresponding to a distinct stable fixed point of $F(\boldsymbol{\psi})$ (or, in terms of the particle description, a distinct local maximum of the potential when such exists). To each such family there corresponds a structurally stable solution, and following the above reasoning, we believe that in each case this structurally stable solution is the slowest member of its family.

VII. LINEAR MINIMUM SPEED ANALYSIS

For the single-mode equations we have been studying, the structural stability condition supports the minimum speed hypothesis. The situation for multiple-mode equations is not as clear, but here again we conjecture that the physically realizable solution of a particular equation is the slowest stable one. Apparently the minimum speed characterization is generic to (at least many types of) semilinear parabolic PDE's. In this section we introduce a method to calculate propagation speeds for these PDE's which exploits the minimum speed condition. Although this method employs only a linear calculation and can therefore not be used for equations whose realizable propagation speeds depend on nonlinear terms, it appears to be at least as powerful as marginal stability theory.

We consider the general N-mode coupled semilinear parabolic PDE,

$$\partial \boldsymbol{\psi}/\partial t = \mathbf{D}\nabla^2 \boldsymbol{\psi} + F(\boldsymbol{\psi}) , \tag{7.1}$$

where $\mathbf{D}$ is an $N \times N$ real matrix. Assuming a traveling-wave solution, and considering the linear (large $\xi \equiv x - ct$) regime, $\boldsymbol{\psi}$ is given by $\mathbf{A}\exp(-k\xi)$, where $\mathbf{A}$ is a constant (i.e., independent of ξ) N vector, and we have

$$(k^2\mathbf{D} + \mathbf{F} - ck)\mathbf{A} = \mathbf{0} , \tag{7.2}$$

with $\mathbf{F}$ being an $N \times N$ real matrix. To solve for c, we need only find the eigenvalues $\lambda_i(k)$ of $k^2\mathbf{D} + \mathbf{F}$. We then have speeds $c_i(k) = \lambda_i(k)/k$. Each function $c_i(k)$ corresponds to a branch of eigenvalues of this matrix and represents the speeds of the associated propagating solutions described by (7.2). At this time, we restrict our attention to only those equations for which all $c_i(k)$ are real for all values of $k > 0$. (k must be positive by assumption.) Assuming the propagating front behavior of (7.1) is described by the linear equation (7.2), we now set out to determine which speed is realized in a physical system.

For the case $N = 1$, we know that the physically observable propagation speed is that of the slowest stable front. In the case that this speed is determined solely by linear order terms, this speed corresponds to the minimum of $c(k)$. Any propagation speed corresponding to a different value of k cannot be realized from initial conditions defined on a compact support. For the general N-mode (linear) case, we conjecture that the analogous statement holds. That is, only if a solution corresponds to the minimum of some branch of speeds $c_i(k)$, can it be realized from physical initial conditions. Furthermore, we assert that although only one of these speeds is realizable, all unrealizable minimum speeds can appear in the form of transient disturbances. Note that if no $c_i(k)$ has a minimum at any value of $k > 0$, or if no minimum has positive value, our conjecture implies that no traveling-wave solution can be realized through the application of physical initial conditions.

Stated more explicitly, our conjecture is that the application of physical initial conditions excites the system in such a way that asymptotically (for the moment, ignoring the role played by the nonlinear stabilizing terms) $\boldsymbol{\psi}(x,t)$ approaches a solution consisting of a superposition of wave forms, each converging to a steady state solution of the linearization of (7.1). These steady state solutions correspond to minima of the functions $c_i(k)$. Given a set of initial conditions, there exists a set of such steady state solutions which are excited in this sense. Then as $t \to \infty$, the wave form converging to the fastest such solution will establish itself ahead of all others, and all slower disturbances will be destroyed by the nonlinear stabilizing terms in the wake of this fastest front. The fastest member among the set of excited solutions will thus be selected. Finally, we conjecture that for "almost all" initial conditions, all minimum speed solutions are excited and thus that the fastest speed among the set of minima is selected.

Let us consider an example. The equations

$$\begin{aligned} \frac{\partial \psi_1}{\partial t} &= \nabla^2\psi_1 + \psi_1 + \tfrac{1}{2}\psi_2 - \psi_1^3 , \\ \frac{\partial \psi_2}{\partial t} &= \nabla^2\psi_2 + \psi_2 + \tfrac{1}{2}\psi_1 - \psi_2^3 , \end{aligned} \tag{7.3}$$

when linearized about zero yield

$$\begin{aligned} \frac{\partial \psi_1}{\partial t} &= \nabla^2\psi_1 + \psi_1 + \tfrac{1}{2}\psi_2 , \\ \frac{\partial \psi_2}{\partial t} &= \nabla^2\psi_2 + \psi_2 + \tfrac{1}{2}\psi_1 . \end{aligned} \tag{7.4}$$

Equation (7.4) has eigenfunctions $(\psi_1,\psi_2) = (\tfrac{1}{2})^{1/2}(1,1)\exp(-k\xi)$ and $(\tfrac{1}{2})^{1/2}(1,-1)\exp(-k\xi)$, with corresponding speeds $k + 3/2k$ and $k + 1/2k$. The minimum speeds occur at $k = \sqrt{\tfrac{3}{2}}$ and $k = \sqrt{\tfrac{1}{2}}$, where $c = \sqrt{6}$ and $c = \sqrt{2}$, respectively. We can write an arbitrary set of initial conditions $(\psi_1^0(x), \psi_2^0(x))$ as $(\tfrac{1}{2})^{1/2}(1,1)h_1(x) + (\tfrac{1}{2})^{1/2}(1,-1)h_2(x)$. Since D is a scalar, in the linear (small ψ_1 and ψ_2) region of the resulting traveling-wave solution of (7.3), symmetric and antisymmetric functions evolve independently. In addition, we assume the functional form of the linear region of any realizable traveling-wave solution of (7.3) is independent of the nonlinear terms for all time. Thus the asymptotic (in time) nature of the linear region should be determined by the fastest linear mode, unless $h_1(x) \equiv 0$. We therefore conclude that if $h_1(x) \not\equiv 0$, the $c = \sqrt{6}$ traveling-wave solution of (7.3) will be realized, but if $h_1(x) \equiv 0$, the $c = \sqrt{2}$ solution will be realized. We checked this conclusion numerically by giving (7.4) many different sets of initial conditions for which $h_1(x)$ was not identically zero, and in each case, the symmetric steady state solution was realized ($\sqrt{6} \approx 2.45$, while the speeds found numerically for all such initial conditions were $c = 2.44$). It should be noted that this was found to be the case even when $h_1(x)$ was as small as $\sim 10^{-5}h_2(x)$. However,

TABLE II. $c_{\text{max-min}}$ represents the speed of the fastest minimum speed traveling-wave solution for each of the four equations in question, (7.5)–(7.8). c_{num} is the propagation speed found by numerically integrating these equations after defining some nonzero set of initial conditions. In each case, the numerical calculation was repeated for several sets of initial conditions, and in each case no dependence of the asymptotic speed on initial conditions was observed. The discrepancies between values of $c_{\text{max-min}}$ and c_{num} was found to decrease continuously with increased numerical precision.

Eq. no.	$c_{\text{max-min}}$	c_{num}
(7.5)	3.88	3.86
(7.6)	5.30	5.29
(7.7)	5.98	5.96
(7.8)	2.45	2.46

when $h_2(x)$ was set identically to zero, the disturbance asymptotically approached the antisymmetric solution ($\sqrt{2}\approx 1.41$, while numerically we found $c=1.42$).

In general, the evolution of $\boldsymbol{\psi}(x,t)$ in a coupled N-mode system is not as simple as that in (7.4), because $\mathbf{D}$ is not, in general, a scalar. However, we believe that as in that case, for any N-mode equation of the type in question, the fastest member of the set of minimum speed solutions is excited by all but a measure zero set of initial conditions. Thus this solution should be the physically realizable one. We checked this claim for the following equations. Results of this study are listed in Table II. In all cases, these results are consistent with our conclusion.

$$\frac{\partial\psi_1}{\partial t}=2\nabla^2\psi_1+\psi_1+\tfrac{1}{2}\psi_2-\psi_1^3\,,\qquad \frac{\partial\psi_2}{\partial t}=\nabla^2\psi_2+3\psi_2+\tfrac{1}{2}\psi_1-\psi_2^3\,. \tag{7.5}$$

$$\frac{\partial\psi_1}{\partial t}=2\nabla^2\psi_1+\nabla^2\psi_2+2\psi_1+\psi_2-\psi_1^3\,,\qquad \frac{\partial\psi_2}{\partial t}=\nabla^2\psi_2+\nabla^2\psi_1+2\psi_2+\tfrac{1}{2}\psi_1-\psi_2^3\,. \tag{7.6}$$

$$\frac{\partial\psi_1}{\partial t}=2\nabla^2\psi_1+2\nabla^2\psi_2+4\psi_1+\psi_2-\psi_1^3\,,\qquad \frac{\partial\psi_2}{\partial t}=\nabla^2\psi_2+\psi_2+\tfrac{1}{2}\psi_1-\psi_2^3\,. \tag{7.7}$$

$$\frac{\partial\psi_1}{\partial t}=\nabla^2\psi_1+\tfrac{1}{2}\nabla^2\psi_2+\psi_1-\psi_1^3\,,\qquad \frac{\partial\psi_2}{\partial t}=2\nabla^2\psi_2-\tfrac{1}{2}\nabla^2\psi_1+\psi_2-\psi_2^3\,. \tag{7.8}$$

VIII. NUMERICAL EXAMPLES

A. Single-mode systems

For AW-type equations, the max-min principle confirms the minimum speed (or critical damping) characterization made in the previous section. However, for non-AW-type equations, there exists no rigorous confirmation of this characterization. We wish to make a check of our predictions in this case as well, and do so below for one particular equation. In this case, through numerical study, we obtain results consistent with the minimum speed characterization implied by the structural stability hypothesis.

We consider the following PDE:

$$\frac{\partial\psi}{\partial t}=\frac{\partial^2\psi}{\partial x^2}+\psi^2-\psi^3\,. \tag{8.1}$$

Because $F'(\psi)|_{\psi=0}=0$, this is not of the AW type. Assuming a traveling-wave solution to (8.1) yields

$$\frac{d^2\varphi}{d\xi^2}+c\frac{d\varphi}{d\xi}+\varphi^2-\varphi^3=0\,. \tag{8.2}$$

Using the method outlined by van Saarloos [18], the slowest stable traveling-wave solution can be determined explicitly. We find $\psi^*(x,t)=\{1+\exp[(x-ct)/\sqrt{2}]\}^{-1}$, where the propagation speed is $c=1/\sqrt{2}\approx 0.7071$. All traveling-wave solutions with speed $c>1/\sqrt{2}$ decay more slowly than an exponential in the $x\to\infty$ limit. We numerically studied (8.1) by perturbing the $\psi\equiv 0$ solution and then integrating to large enough time that a traveling-wave solution was realized. The speed of the selected solution of (8.1) was thus found to be $c=0.7070$. We then numerically determined the critical value of c for (8.2), finding $c=0.7071$. The minimum speed characterization is therefore confirmed.

Though to this point our analysis of single-mode equations has been limited to those of the Fisher form, we believe that the minimum speed characterization of selected solutions applies more generally. It is easy to imagine equations more complicated than (2.1) which can nonetheless be cast into the particle-on-a-hill form and for which the arguments of the previous section should remain valid. In particular, this should be the case for many ambiguous PDE's whose corresponding particle equations of motion assume (or at least approach) the form $\dot{p}=-cp+\mathcal{F}(q)$ near the "bottom of the hill," $q=0$. We now consider one such PDE and obtain results consistent with this conclusion.

Due to its $|\partial\psi/\partial x|$ dependence, the equation

$$\frac{\partial\psi}{\partial t}=\frac{\partial^2\psi}{\partial x^2}+\psi\left[1+a\Theta\left[\left|\frac{\partial\psi}{\partial x}\right|-b\right]\right]\Theta(1-|\psi|)\,, \tag{8.3}$$

where a and b are positive constants, is not of the Fisher form. (Here, Θ is the step function. We should note that the singularity it introduces is not essential.) Assuming a traveling-wave gives

$$\frac{d^2\varphi}{d\xi^2}+c\frac{d\varphi}{d\xi}+\varphi\left[1+a\Theta\left[\left|\frac{d\varphi}{d\xi}\right|-b\right]\right]\Theta(1-|\varphi|)=0\,, \tag{8.4}$$

Again, through numerical computation, we found speeds of selected traveling-wave solutions to (8.3), as well as the critical value of c in (8.4) for several values of a and b. The results of this study are shown in Table III. In all cases, these are consistent with our hypothesis: the selected solution is that which corresponds to the critically damped particle trajectory.

TABLE III. For each of the values of a and b shown, we determined the propagation speed of the corresponding traveling wave, $c_{\rm PDE}$, by numerically integrating (8.3). We then numerically determined the critical value of c for (8.4), $c_{\rm crit}$. In each case, the precision to which the two values agree was apparently only limited by the coarseness of the discretization used in the numerical calculations.

a	b	$c_{\rm PDE}$	$c_{\rm crit}$
4	0.1	3.908	3.908
4	0.3	2.674	2.672
6	0.1	4.648	4.648
6	0.3	2.676	2.672

B. Two-mode systems

1. Linearly decoupled case

For systems with $N>1$, as mentioned previously, the particle trajectory is determined by both the value of c and the particle's initial direction. We set out to check our hypothesis that the trajectory $(\mathbf{n}_0, c_0)$ corresponding to the selected solution of (6.1) is critically damped. For simplicity, we chose a PDE whose selected solution is such that no mode changes sign at any point. In this case, to confirm our hypothesis, we must only demonstrate that there exist trajectories $(\mathbf{n}', c')$ arbitrarily close to $(\mathbf{n}_0, c_0)$ for which all coordinates change sign at least once. In fact, we obtained results consistent with the existence of trajectories $(\mathbf{n}_0, c')$ arbitrarily close to $(\mathbf{n}_0, c_0)$ satisfying this condition.

Fixing $\mathbf{n}=\mathbf{n}_0$, we saught the "critical" values of c for each q_i. c_i is a critical value for q_i if for any interval Δc containing values on either side of c_i, there exist values for which q_i changes sign at least once and values for which q_i never changes sign. For the example considered here, we found that for each mode there exists only one critical value of c. Proceeding by first numerically generating traveling-wave solutions to the PDE in question, we then used these solutions to supply initial conditions for the corresponding particles in the classical mechanical analogues. In each case, we selected a point near the back of the wave front, where each mode is near its equilibrium value. Then recalling the identifications $\varphi_i \to q_i$ and $d\varphi_i/dx \to p_i$, we obtained a set of initial conditions for the particle near $\mathbf{q}_1$. [Note that in this case, $\mathbf{q}_1 \neq (1,1)$.] With these initial conditions fixed, we then determined the critical values of c for each mode independently. Our hypothesis is that all values so determined should be equal to each other and to the propagation speed found by solving the PDE.

We investigated the following PDE:

$$\begin{aligned}\frac{\partial \psi_1}{\partial t} &= 2\frac{\partial^2 \psi_1}{\partial x^2} + \psi_1 + s\psi_1\psi_2 - \tfrac{1}{2}\psi_1^3 - \psi_1\psi_2^2 ,\\ \frac{\partial \psi_2}{\partial t} &= \frac{\partial^2 \psi_2}{\partial x^2} + \psi_2 + s\psi_2\psi_1 - \tfrac{1}{2}\psi_2^3 - \psi_2\psi_1^2 .\end{aligned} \tag{8.5}$$

There is perhaps no simpler $N>1$ realization of (6.1) for which traveling-wave solutions exist. Of course, we considered only values of s large enough that the realizable solutions consisted of two-mode invasions. The results of our study are shown in Fig. 1 and Table IV. In all cases, these results support our hypothesis.

2. Linearly coupled case

Although we have not been able to demonstrate a characterization of the selected solutions of linearly coupled multiple-mode equations, we believe that such solutions

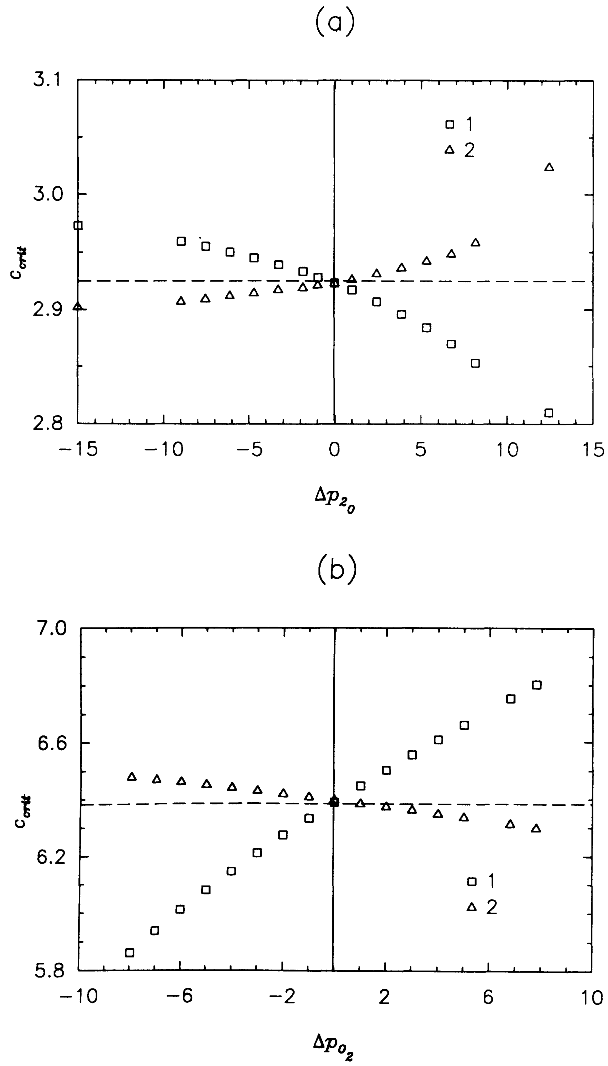

FIG. 1. To determine the values of $c_{\rm crit}$ shown in Table IV, we obtained particle initial conditions $(q_{10}, q_{20}, p_{10}, p_{20})$ from the wave fronts generated by numerically integrating (8.5). Then, fixing the initial values of q_1, q_2, and p_1 at q_{10}, q_{20}, and p_{10}, we determined values of $c_{\rm crit}$ corresponding to each mode for initial conditions of p_2 on either side of p_{20}. (a) depicts results for the case $s=3$, and (b) those for $s=9$. The dashed lines correspond to the speeds found by integrating (8.5). Δp_{20} is the uncertainty in determining p_{20} from the numerically generated wave fronts. Similar results were obtained by holding the initial value of p_2 fixed at p_{20} and varying the initial value of p_1. In each case, the results are consistent with our hypothesis: for the trajectory corresponding to the selected solution, each mode is critically damped.

TABLE IV. We first numerically integrated (8.5) to large enough times that traveling-wave solutions were realized. The propagation speeds so determined are represented here by c_{PDE}. Using these traveling-wave solutions, we obtained initial conditions for the "particles" whose trajectories are described by ODE's corresponding to (8.5). From these initial conditions, we were able to determine independently critical values of c for each mode. Those found for ψ_1 and ψ_2 are designated here by $c_{1\mathrm{crit}}$ and $c_{2\mathrm{crit}}$. For each s we found disagreement between all values monotonically decreasing with increasingly fine discretizations.

s	$c_{1\mathrm{crit}}$	$c_{2\mathrm{crit}}$	c_{PDE}
3	2.923	2.923	2.925
4	3.396	3.397	3.394
5	3.943	3.943	3.939
6	4.527	4.528	4.525
7	5.135	5.136	5.132
8	5.757	5.760	5.754
9	6.388	6.395	6.382

too must be structurally stable. We checked this point numerically for (7.5).

Let us define F_1 and F_2 such that (7.5) can be written as

$$\partial\psi_1/\partial t = 2\nabla^2\psi_1 + F_1(\psi_1,\psi_2)\ ,$$
$$\partial\psi_2/\partial t = \nabla^2\psi_2 + F_2(\psi_1,\psi_2)\ . \tag{8.6}$$

We assume that the perturbation represented by $F_1 \to F_1 + \delta F_1$ and $F_2 \to F_2 + \delta F_2$, where $\delta F_i = -10\psi_i$ if $\psi_i < \delta$ and 0 otherwise, becomes physically small as $\delta \to 0$. Then, if a traveling-wave solution of (8.6) with speed c is structurally stable, the speed of the selected solution of the perturbed equation must converge to c as $\delta \to 0$. We numerically determined propagation speeds for selected solutions of perturbed versions of (8.6) with several values of δ. The results of this study, shown in Table V, support our structural stability hypothesis.

We next studied a physically unrealizable solution of (8.6). We were able to produce such a solution by choosing two small positive values ϵ_1 and ϵ_2, and forcing the value of x at which both $\psi_1 = \epsilon_1$ and $\psi_2 = \epsilon_2$ to move at speed $c = 10$. Let us call this system "tip driven" and the system considered above "bulk driven." We chose $\epsilon_1 = 0.248 \times 10^{-11}$ and $\epsilon_2 = 10^{-11}$. With these values, the eigenfunction of the linear equation coresponding to (8.6) for the traveling-wave solution with $c = 10$ is given by $\mathrm{const} \times (\epsilon_1, \epsilon_2)\exp(-kx)$, with $k = 0.323$. We then computed the speed of the resulting front by watching the point at which $\psi_i = 0.01$. Not surprisingly, this value was 10. However, when we applied perturbations to the tip driven system identical to those applied to the bulk driven system, in each case, the propagation speed computed was also identical to that found for the bulk driven system. The faster front is therefore destroyed by these perturbations, and we conclude that it is structurally unstable.

TABLE V. The value of δ determines the size of the "bump" at the origin. For all nonzero values of δ, the speeds listed here were observed for both the bulk driven and tip driven systems. The speed for the $\delta = 0$ case is that observed in the bulk driven system.

δ	c
10^{-5}	3.68
10^{-6}	3.73
10^{-7}	3.77
10^{-8}	3.79
10^{-9}	3.81
10^{-12}	3.83
0	3.86

IX. EQUATIONS WITH MULTIPLE PHYSICALLY REALIZABLE SOLUTIONS

For each of the PDE's we have investigated to this point, there exists a unique structurally stable selected solution. This implies that the steady state behavior of the physical system modeled by any one of these PDE's is independent of its initial conditions. For the $N = 2$ case considered above, in addition to the selected two-mode solution, there are structurally stable single-mode solutions obtainable by setting one mode to zero. These solutions, however, are unstable with respect to small perturbations of the null modes and therefore unrealizable. There is certainly no reason, however, that such must be the case for all systems exhibiting front propagation. In this section, we consider systems for which there exists no unique selected solution, but for which the steady state behavior differs for different sets of physically realizable initial conditions [28]. We will begin by studying three relatively simple equations and then consider a more complicated, but more directly physically motivated system. For the latter, we will demonstrate how the stability of a certain structurally stable solution with respect to fluctuations changes with temperature and how, as a result, the number of realizable solutions changes from one to two.

A. Simple examples

Consider the following system of coupled equations:

$$\frac{\partial\psi_1}{\partial t} = \frac{\partial^2\psi_1}{\partial x^2} + [\psi_1 - a_1\psi_1\Theta(|\psi_2| - b)]\Theta(1 - |\psi_1|)\ ,$$
$$\frac{\partial\psi_2}{\partial t} = \frac{\partial^2\psi_2}{\partial x^2} + [\psi_2 - a_2\psi_2\Theta(|\psi_1| - b)]\Theta(1 - |\psi_2|)\ , \tag{9.1}$$

where $a_1, a_2 > 0$ and $0 < b < 1$. We should point out that the nondifferentiability of (9.1) is not crucial. That is, by slightly modifying this equation, we can produce a smooth equation exhibiting the same general features. Here, the initial conditions and the subsequent race to b determines the asymptotic behavior. That mode which is favored by the initial conditions and attains the value b ahead of the other will lead the invasion into the zero solution. The steady state solutions are the following:

$$\psi_i = \begin{cases} (1+x-c_i t)\exp[-(x-c_i t)] , & x-c_i t>0 , \\ 1, & x-c_i t \le 0 , \end{cases} \tag{9.2}$$

with $c_i=2$ and

$$\psi_j = \begin{cases} [1+\sqrt{1-a_j}(x-c_j t)]\exp[-\sqrt{1-a_j}(x-c_j t)] , & x-c_j t>0 , \\ 1 , & x-c_j t \le 0 , \end{cases} \tag{9.3}$$

with $c_j=2\sqrt{1-a_j}$, for the leading and trailing modes, respectively.

As a somewhat more interesting example, we consider the following set of coupled equations:

$$\begin{aligned} \frac{\partial \psi_1}{\partial t} &= \frac{\partial^2 \psi_1}{\partial x^2} + \psi_1 + d\psi_1|\psi_1| - \tfrac{1}{2}\psi_1\psi_2^2 - \psi_1^3 , \\ \frac{\partial \psi_2}{\partial t} &= 2\frac{\partial^2 \psi_2}{\partial x^2} + \psi_2 - \tfrac{1}{2}\psi_2\psi_1^2 - \psi_2^3 . \end{aligned} \tag{9.4}$$

As above, nonanalyticity here is not crucial. Using ψ_1^2 instead of $\psi_1|\psi_1|$ would simply limit the following discussion to fronts for which ψ_1 assumes only positive values. The coupling terms here serve to introduce a competition between the two modes. If one mode is somehow able to establish itself ahead of the other, it tends to suppress the trailing mode. We thus recognize the mechanism by which the initial conditions can gain great importance in determining the asymptotic behavior. However, the asymmetry among the diffusion terms and the existence of the $d\psi_1^2$ term present the possibility that, depending on the value of d, one or the other mode will have an "unfair" advantage, and independently of the initial conditions this mode will lead the selected invasion.

A simple analysis reveals that there exists $d_1 \in (0, 1+3/\sqrt{2})$ such that for all $d<d_1$ the selected propagating front is one for which ψ_2 leads. Similarly, there exists $d_2 \in (1+3/\sqrt{2}, \sqrt{2}+3\sqrt{3/2})$ such that for all $d>d_2$, the selected propagating front is one for which ψ_1. (In fact, in this case, $\psi_2 \equiv 0$.) Numerically, we have found that $d_1 \approx 2.8$ and $d_2 \approx 3.4$. Thus, for $2.8<d<3.4$, the asymptotic behavior of the system is determined by the initial conditions. Outside this region, this behavior is independent of the initial conditions (assuming neither mode is identically zero). In Fig. 2, the speed of each mode is plotted as a function of d.

Equations (9.1) and (9.4) are both examples of the type of equation studied in Sec. VI. Thus in both cases, the structural stability hypothesis implies that the initial conditions choose among structurally stable and therefore critically damped solutions. For both equations, the stable fixed points to which the leading invasions of the physically realizable solutions converge behind their fronts are each nonzero for only one mode. The equations governing these leading modes are all AW-type PDE's, for which the validity of the structural stability hypothesis and the minimum speed characterization are rigorously established. The physically realizable solutions of (9.1) and (9.4) are therefore indeed the slowest members of distinct families of stable solutions (i.e., the stable solutions of the AW-type equations in question), as conjectured.

For the two examples thus far considered, competition between mutually destructive modes leads to a selection between physically realizable solutions. In this case, the mutually destructive nature of the modes precludes the existence of a stable traveling-wave solution representing the simultaneous invasion of both modes. The initial conditions therefore select one mode at the expense of the

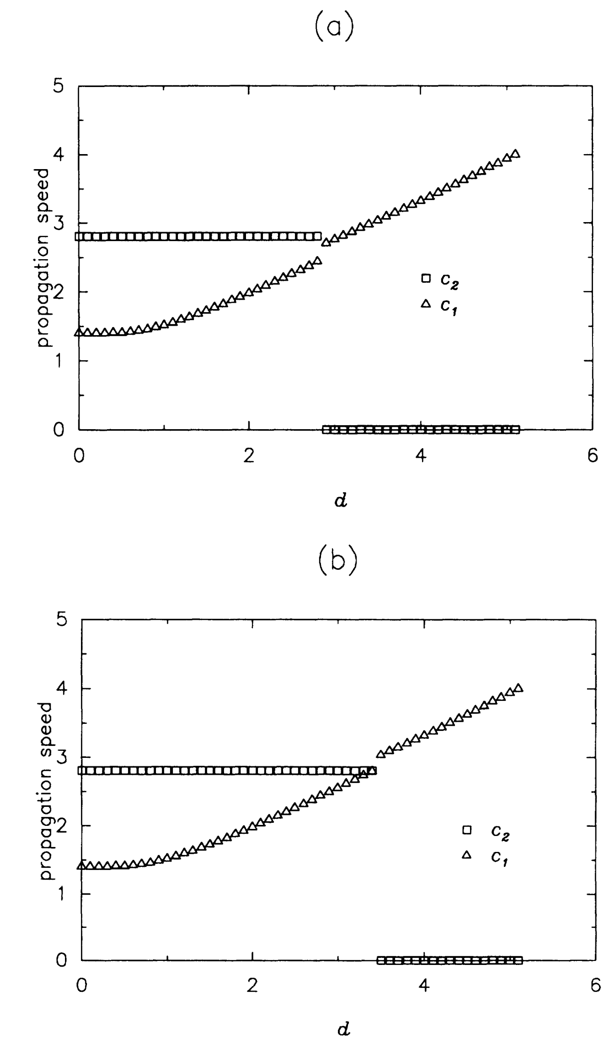

FIG. 2. c_1 represents the propagation speed of ψ_1 and c_2 that of ψ_2 for (9.4). (a) and (b) depict the cases in which the initial conditions favor ψ_1 and ψ_2, respectively.

other. We now consider an equation for which the asymptotic behavior is not determined by such a mode-mode competition, but for which nonetheless there exist multiple physically realizable solutions. In this case, the initial conditions select among physically realizable solutions displaying varying degrees of cooperative behavior. The selection is not between competing modes, but among ways of exploiting the (sometimes) cooperative interaction between modes.

We consider the following:

$$\frac{\partial\psi_1}{\partial t}=\frac{\partial^2\psi_1}{\partial x^2}+\psi_1+\psi_1\psi_2+s\psi_1\psi_2^2-\tfrac{1}{4}\psi_1^5-\psi_1\psi_2^4\,,$$
$$\frac{\partial\psi_2}{\partial t}=\frac{\partial^2\psi_2}{\partial x^2}+\psi_2+\psi_1\psi_2+s\psi_2\psi_1^2-\tfrac{1}{4}\psi_2^5-\psi_2\psi_1^4\,. \tag{9.5}$$

The second order terms here introduce an interesting effect. Depending on the signs of the two modes, these terms can encourage or discourage their growth. In addition, if at any time a given mode assumes values of strictly one sign, it must remain this way for all later times. It is thus apparent how the initial conditions can play a crucial role in determining the asymptotic behavior.

For all values of s, there are three distinct physically realizable traveling-wave solutions to (9.5), one for which both modes assume only positive values, one for which one mode assumes only positive values, while the other assumes only negative values (we do not distinguish between the two degenerate such solutions), and one for which both modes assume only negative values. In all cases, the invasion consists of a single traveling wave whose front region is composed of nonzero contributions of each mode. Propagation speeds of these solutions are plotted in Fig. 3.

Equation (9.5) is again a member of the class of equations studied in Sec. VI, and thus again, the structural stability hypothesis implies that all of its physically realizable solutions are critically damped. All stable traveling-wave solutions converging to the $(+,+)$ fixed point satisfy $\psi_1=\psi_2$. For this reason, each such solution satisfies the same AW-type equation. For those stable

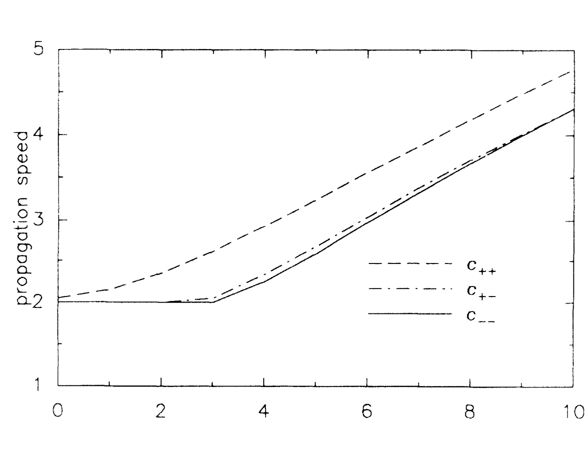

FIG. 3. c_{++}, $c_{+,-}$, and c_{--} are the speeds of the solutions to (9.5) for which both modes are positive, one mode is positive and one negative, and both modes are negative, respectively.

solutions converging to the $(-,-)$ fixed point, the same statement holds. Thus the physically realizable $(+,+)$ and $(-,-)$ solutions are also the slowest stable solutions of their respective families. The stable solutions corresponding to the $(+,-)$ fixed point are such that asymptotically their two modes differ by only a constant (negative) factor. Obviously there exist such solutions which are faster than the physically realizable one (those which asymptotically decay to zero sufficiently slowly). Then, as discussed near the end of Sec. VI, we believe that any slower propagating front corresponding to the $(+,-)$ fixed point should be unstable with respect to the bulk invasion which drives all physical $(+,-)$ initial conditions to the realizable solution. We therefore believe that in this case too, the physically realizable solution is the slowest stable propagating front.

The three systems considered above are somewhat artificial in the sense that they lack physical motivation, and in each case, the reasons for the critical dependence on the initial conditions is not at all mysterious. We now turn our attention to a physically motivated system in which similar but more interesting behavior is exhibited.

B. Diblock copolymer system

We consider a coupled set of equations describing front propagation in a 2-space diblock copolymer (DBCP) system. For a detailed description of the derivation of these equations as well as discussion of experimental feasibility, see [29]. This set of equations describes the motion of modes of ordered structure as they invade a disordered region. We consider here the case in which the disordered region is unstable with respect to ordering. In this case, the most important features of the invasion process can be understood by studying the following reduced set of coupled equations:

$$\begin{aligned}
\frac{\partial W_{1,1}}{\partial t}&=\nabla^2 W_{1,1}+W_{1,1}-sW_{1,2}W_{1,3}-\tfrac{3}{4}W_{1,1}\sum_{i,j}(1-\tfrac{1}{2}\delta_{1-i}\delta_{1-j})W_{i,j}^2\,,\\
\frac{\partial W_{1,2}}{\partial t}&=\tfrac{1}{4}\nabla^2 W_{1,2}+W_{1,2}-sW_{1,1}W_{1,3}-\tfrac{3}{4}W_{1,2}\sum_{i,j}(1-\tfrac{1}{2}\delta_{1-i}\delta_{2-j})W_{i,j}^2\,,\\
\frac{\partial W_{1,3}}{\partial t}&=\tfrac{1}{4}\nabla^2 W_{1,3}+W_{1,3}-sW_{1,1}W_{1,2}-\tfrac{3}{4}W_{1,3}\sum_{i,j}(1-\tfrac{1}{2}\delta_{1-i}\delta_{3-j})W_{i,j}^2\,,\\
\frac{\partial W_{2,1}}{\partial t}&=\tfrac{3}{4}\nabla^2 W_{2,1}+W_{2,1}-sW_{2,2}W_{2,3}-\tfrac{3}{4}W_{2,1}\sum_{i,j}(1-\tfrac{1}{2}\delta_{2-i}\delta_{1-j})W_{ij}^2\,,\\
\frac{\partial W_{2,2}}{\partial t}&=\tfrac{3}{4}\nabla^2 W_{2,2}+W_{2,2}-sW_{2,1}W_{2,3}-\tfrac{3}{4}W_{2,2}\sum_{i,j}(1-\tfrac{1}{2}\delta_{2-i}\delta_{2-j})W_{i,j}^2\,,\\
\frac{\partial W_{2,3}}{\partial t}&=W_{2,3}-sW_{2,1}W_{2,2}-\tfrac{3}{4}W_{2,3}\sum_{i,j}\left[1-\tfrac{1}{2}\delta_{2-i}\delta_{3-j}\right]W_{i,j}^2\,.
\end{aligned} \tag{9.6}$$

Here s represents a reduced temperature. These equations govern the triplets $\tilde{W}_1=\{W_{1,1},W_{1,2},W_{1,3}\}$ and $\tilde{W}_2=\{W_{2,1},W_{2,2},W_{2,3}\}$ describing two triangular ordered patterns whose orientations differ by 90°.

Note that (9.6) is again an example of the type of equation studied in Sec. VI, and therefore, by the structural stability hypothesis, all of its physically realizable solutions correspond to critically damped trajectories. For each of these solutions, assertions almost identical to those made about the $(+,-)$ solution of (9.5) again apply. We therefore conjecture that these solutions represent the slowest stable members of their families.

Depending on s, the equilibrium of this system is either a lamellar phase, consisting of only one nonzero mode, or a triangular phase, consisting of a single nonzero triplet of modes. Thus, given s and a set of initial conditions, there is a competition between modes or between triplets, and eventually only one mode or triplet survives and evolves toward a steady state traveling-wave solution. In fact, we have found that there are several distinct regimes, defined by s, in which qualitatively different front propagation behavior is exhibited. This behavior becomes progressively more complicated as s increase from 0. In this paper we will consider only values of s large enough that the equilibrium phase is the triangular and that each physically realizable propagating front solution consists of a traveling-wave invasion of a triplet of modes.

Propagation speeds as a function of s for $\tilde{W}_1,c_1$, and $\tilde{W}_2,c_2$, in this regime are shown in Fig. 4. For $s<4.2, c_1>c_2$. Throughout this regime, c_1 is independent of s and determined by only the linear order terms in (9.6). In particular, $c_1=2$ is determined by the linearized equation describing the evolution of $W_{1,1}$ for small amplitudes, $\partial W_{1,1}/\partial t=\nabla^2 W_{1,1}+W_{1,1}$. Here, c_1 is thus identical to the speed of small, localized disturbances of $W_{1,1}$. Since in this regime $c_2<2$, $\tilde{W}_2$ is unstable with respect to such disturbances, and the unique selected solution therefore consists of an invasion of $\tilde{W}_1$.

Although c_1 is determined by only linear order terms for $s<4.2$, the same is not true for c_2. Near $s=2.2$, c_2 begins to acquire an s dependence and is no longer determined solely by the linear order terms in (9.6). Due to this nonlinear effect, near $s=4.2$, c_2 becomes larger than 2. As a result, $\tilde{W}_2$ is now stable with respect to perturbation by $W_{1,1}$, and the mechanism responsible for selecting $\tilde{W}_1$ has thus been lost. There is no longer a unique selected solution. In this regime, to both orientations of ordered structure, there corresponds a physically observable solution representing the invasion of a triangular pattern.

Because small fluctuations are no longer effective in selecting an orientation, the outcome of a given experiment is determined by the initial conditions alone. Clearly, these can be chosen so that the propagating front solution corresponding to either orientation is realized.

The behavior observed for the DBCP system should be quite general for systems in which an ordered pattern with an orientational degree of freedom (i.e., an ordered pattern in >1 dimensions) propagates into a disordered region. If the fastest speed present is that corresponding to small disturbances of a particular mode of ordered structure, fluctuations in the leading edge of the front will allow the linear order terms to select their "favorite" orientation. When the speeds of the fully developed fronts are larger than that of these small disturbances, the fluctuations can no longer force such a selection, and assuming the nonlinear terms favor no orientation, there will be no unique selected solution.

Although it may be possible to construct a system for which this plausibility argument does not hold, for those systems, such as the DBCP, in which couplings between orientations lead to only stabilizing terms in the equations of motion, the argument clearly holds. Simply stated, in such cases, if a perturbation cannot grow in the leading edge where the front has effectively zero amplitude, it also cannot grow in a region of finite amplitude. Thus for such systems, if there is a unique selected solution, it is singled out from the other structurally stable solutions by a mechanism whose origin lies in the linear order terms. If the linear terms are able to produce no such mechanism, there will be no unique selected solution.

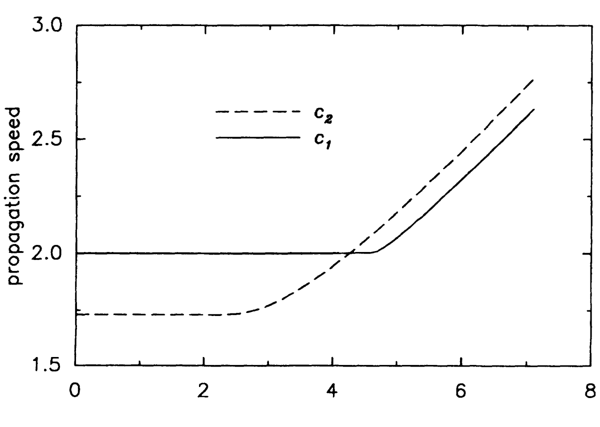

FIG. 4. c_1 and c_2 represent the propagation speeds of $\tilde{W}_1$ and $\tilde{W}_2$. Near $s=4.2$, c_2 becomes larger than c_1. c_2 begins to acquire an s dependence near $s=2.2$. c_1 does so near $s=4.7$.

X. SUMMARY AND CONCLUDING REMARKS

The basic idea of this paper is that a good model of reproducibly observable phenomena must be structurally stable, i.e., the physical predictions provided by the model must be stable against small modifications of the model. This follows naturally from the fact that we cannot know every detail of a given physical system and therefore cannot repeatedly prepare it in an identical way. There are, of course, numerous unstable systems in nature whose behavior can suffer very large change in response to very small perturbations. A good model of such a system must be able to capture this instability, and thus should be structurally *unstable*. In this paper, we have limited our attention to systems exhibiting reproducibly observable phenomena. The idea of structural stability was introduced by Andronov and Pontrjagin. Although Andronov and Pontrjagin require structural stability of all aspects of the model, we require this only of the behavior corresponding to reproducibly observable physical properties. There may be aspects of a given model which do not correspond to any physically realizable phenomena, or which correspond to unstable phenomena in which we have no interest. Certainly, these aspects of the model need not be structurally stable.

In this paper we have applied the structural stability idea outlined above to a special problem, the selection of front propagation speed in semilinear parabolic PDE's. We assume that these PDE's are reasonable models of front propagation in physical systems. Then, the reproducibly observable frontal speeds of such systems should correspond to the structurally stable solutions of these models. Incidentally, the selection problem of the growth speed of needle crystals may appear very similar to the problems we consider here, but there is a crucial distinction. In the needle crystal case, as must be concluded when one considers the unstable nature of the well known Ivanzov solution, the problem is not that of determining physically meaningful solutions, but of constructing a physically reasonable model.

The spirit of structural stability is very close to that of renormalization group (RG) theory, as emphasized in [30] and [31]. RG can be interpreted as a method to extract the system behavior insensitive to system details. In other words, RG is a method of selecting structurally stable features and/or solutions. In our problem, we may say each propagating front solution represents a fixed point of the time evolution of the PDE. This time evolution may be interpreted as a renormalization group procedure [32]. We require the p-smallness condition in order to, in the RG jargon, avoid relevant perturbations which change the nature of the problem as, for example, a uniaxial anisotropy changes the Heisenberg model. Some RG ideas have already been applied to calculate the front speed for certain equations [10].

We have found that for single-mode semilinear parabolic PDE's, the structurally stable propagating front is the slowest stable one (using the ordinary sense of "stability"). According to Aronson and Weinberger [12], the slowest stable front is the observable one for a special class of semilinear parabolic equations we call the AW type. Thus our structural stability hypothesis correctly identifies the observable front in these cases. In other cases, we do not have rigorous proof, but numerical examples support our hypothesis.

We have not been able to characterize structurally stable fronts for all multiple-mode semilinear parabolic PDE's, but when these equations exhibit no linear coupling between modes, we have shown that a structurally stable front corresponds to a critically damped orbit in the mechanical analogy, exactly as in the single-mode case. Numerical results supporting this critical damping characterization were found. These structurally stable fronts are not necessarily unique, and in some cases, more than one of them can be realized. In such cases, the observed front is selected from the set of physically realizable solutions by the initial conditions. We believe that each of these structurally stable solutions represents the slowest member of a distinct family of solutions. Several examples supporting this hypothesis have been given.

For single-mode equations, the reason for the structural stability of the observed front is easily understood when one considers what we have identified as the fundamental feature distinguishing the selected solution from all other stable solutions. For any given PDE of the form (2.1) with the AW condition (and very likely, for a much wider class of PDE's), there corresponds a "bulk speed." If unconstrained by the form of its asymptotic region, a solution of such a PDE will be given toward a unique steady state by its bulk. This steady state is the selected traveling-wave solution. All other stable traveling-wave solutions are "tip determined." Only when the form of the asymptotic region of a solution is able to prevent the bulk from driving the system can such a steady state be realized. For these reasons, a small perturbation applied to $F(\psi)$ near $\psi=0$ is able to destroy all but the selected solution. Since a similar statement holds for multiple-mode equations, we believe that the selected solutions to equations of the form (6.1) are set apart from unobservable solutions by the same bulk-tip distinction.

We have studied only time and space independent uniform perturbations. Of course, to faithfully model actual physical systems, we should expect a space-time noise term to be added as a structural perturbation. The solution which can survive this perturbation, then, will be observable. This should be the ultimate version of the structural stability hypothesis, and indeed such a study already exists for dynamical systems (see, for example, the work of Kifer [33], anticipated by Oono and Takahashi [34] in the case of one-dimensional maps). We expect, however, that quite often, and particularly for the equations we have considered, solutions which are structurally stable against time and space independent perturbations will also survive this stochastic torture. Here again, however, as the case of the modification of F considered in the main body of this paper, we must restrict the class of noise. We consider only p-small noise for which Q_η is uniformly small in space and time. Consider the fuse analogy. Noise will produce locally p-small perturbations, so we can imagine a random, continual sprinkling of water over the fuse. Of course, in this case, the water must evaporate even far from the front (i.e., at low temperature) in order to prevent the fuse from becoming so wet that the perturbation is no longer small. If the sprinkling of water is sufficiently strong, then it is intuitively obvious that a solution whose average (in time) speed is larger than c^* cannot propagate on this stained fuse; only the bulk can dry the random wet patches quickly enough to ensure the propagation of the flame. In an intuitive sense, there is little difference between this picture and that considered above in which the film is covered uniformly with a time-independent film of water. In the present case, if we coarse-grain in time, the randomly stained fuse appears uniformly damp. We conclude that the p-small random perturbation selects what we have expected from solutions which are produced from any set of smooth initial conditions.

Although this observation is almost trivial, it has an interesting consequence. We can ask the probability of observing the time averaged propagation speed $\langle c \rangle_T$ for the time span $[0,T]$. We can then ask the probability

$P(\langle c \rangle_T \in U)$ of finding this empirical speed in a given set U of speeds for sufficiently large T. It is very likely that this satisfies the large deviation principle [35] (with respect to T). This implies that there is a variational principle which selects the most probable propagation speed. This solution is the structurally stable one according to our consideration. Of course we can also consider the large deviation with respect to noise amplitude. This should lead to a similar variational principle.

It is an interesting question to ask what happens if we lift the p-smallness condition from the stochastic perturbation. In the fuse analogy, this corresponds to sprinkling water and explosive randomly over the fuse. Even if water could kill the leading edge, the following approach of the bulk could set off the explosive burning to produce a fast leading edge again. Hence, the situation is likely very complicated.

ACKNOWLEDGMENTS

We would like to thank L.-Y. Chen and N. Goldenfeld for their interest in the problem. This work has been supported in part by the National Science Foundation through Grant No. DMR 90-15791. Computational aspects have been supported in part by National Science Foundation Grant No. DMR 89-20538 administered by the University of Illinois Materials Research Laboratory. G.C.P. acknowledges support from the Japanese Society for the Promotion of Science.

APPENDIX

Following Langer and Müller-Krumbhaar [36], a front is considered stable, unstable or marginally stable with respect to a given perturbation if, when viewed from the frame moving with the front, this perturbation shrinks, grows, or remains stationary. More precisely, we consider the traveling-wave solution $\varphi_0(\xi)$, where $\xi = x - ct$. At $t=0$ we apply a perturbation $\delta\varphi(\xi,t)$ and study $\varphi(\xi,t) = \varphi_0(\xi) + \delta\varphi(\xi,t)$. If on each bounded interval $\lim_{t\to\infty}\varphi(\xi,t) = \varphi_0(\xi)$, $\varphi_0(\xi)$ is stable with respect to $\delta\varphi(\xi,t)$. If there exists a finite γ such that on each bounded interval $\lim_{t\to\infty}\varphi(\xi,t) = \varphi_0(\xi+\gamma)$, $\varphi_0(\xi)$ is marginally stable with respect to $\delta\varphi(\xi,t)$. If no such γ exists, or if this limit does not exist, $\varphi_0(\xi)$ is unstable with respect to $\delta\varphi(\xi,t)$. (Generalization to multiple-modes is trivial.)

In general, even if the function $\varphi_0(\xi)$ is known, stability cannot be determined through a linear analysis. We will now study an example which illustrates this point.

Consider the system

$$\frac{\partial\psi_1}{\partial t} = \frac{\partial^2\psi_1}{\partial x^2} + \psi_1 + d\,\psi_1^2\psi_2^2 - \psi_1^5 ,$$
$$\frac{\partial\psi_2}{\partial t} = \frac{1}{2}\frac{\partial^2\psi_2}{\partial x^2} + \psi_2 + d\,\psi_2^2\psi_1^2 - \psi_2^5 . \tag{A1}$$

For all values of s, there is a structurally stable traveling-wave solution to (A1) satisfying $\partial\psi_1/\partial t = \partial^2\psi_1/\partial x^2 + \psi_1 - \psi_1^5$; $\psi_2 \equiv 0$. We will call this solution $\tilde{\psi}_1(x,t)$. For $d < 2.1$, $\tilde{\psi}_1(x,t)$ is stable with respect to perturbation by ψ_2 and therefore physically realizable, while for larger values it is unstable. However, the propagation speed of $\tilde{\psi}_1(x,t)$ is 2, while the linear propagation speed to which a small disturbance of ψ_2 will tend at sufficiently large times [37] is $\sqrt{2}$, independent of d. Thus although for $d > 2.1$, $\tilde{\psi}_1(x,t)$ is unstable with respect to perturbation by ψ_2, it is linearly stable for all d. For values of $d > 2.1$, a small disturbance of ψ_2 will be initially unable to keep up with $\tilde{\psi}_1(x,t)$, and it will be seen to shrink when observed from the frame moving with speed 2. Viewed from the lab frame, however, its amplitude will grow in accordance with the linear equation $\partial\psi_2/\partial t = \frac{1}{2}\partial^2\psi_2/\partial x^2 + \psi_2$. As this growth continues, eventually the $d\psi_2^2\psi_1^2$ term will become non-negligible, and the disturbance will begin to gain speed. Eventually it will reach speeds larger than 2 and will start catching up to the leading front. For values of d between approximately 2.1 and 2.3, as the trailing front reaches the leading front, it will begin to slow down, and the system will approach a final steady state consisting of a single two-mode traveling-wave propagating with speed $c=2$. For values of $d > 2.3$, the trailing front will not slow to the original speed of the leading front, but will instead begin to push the leading front through the $d\psi_1^2\psi_2^2$ term, and the system will evolve toward a two-mode traveling-wave propagating at some speed $c > 2$.

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$$\sup_{\eta>0}(\sup_{\psi\in(0,\eta]}\{[F(\psi)+\delta F(\psi)]/\psi\} - \sup_{\psi\in(0,\eta]}[F(\psi)/\psi])$$

is less than the same small number.
[21] Strictly speaking, the ambiguity of a PDE depends on the designation of boundary conditions, that is, the values to which $\psi(x,0)$ converges at $x=\pm\infty$. Since we consider only traveling-wave solutions, we are limited to stationary boundary conditions. Therefore $\lim_{x\to+\infty}\psi(x,t)=a_1$, and $\lim_{x\to-\infty}\psi(x,t)=a_2$, where the values a_1 and a_2 correspond to stationary solutions of the PDE. The unperturbed PDE's we study will all possess the unstable stationary solution $\psi\equiv 0$ (or $\psi_i\equiv 0$ for all i in multiple-mode examples) and traveling-wave solutions representing the invasion of fronts into this solution. We wish to examine these traveling-wave solutions and will therefore always consider boundary conditions for these PDE's which converge to zero in one direction (arbitrarily choose the $+$ direction). We will restrict the set of allowed perturbed PDE's to those which are unambiguous with respect to stationary boundary conditions satisfying the same condition. Thus all allowable perturbations must satisfy $\delta F(\psi=0)=0$. We could consider perturbed PDE's which are unambiguous with respect to some other stationary boundary conditions and solutions satisfying these. [For example, the perturbed PDE's could be chosen to be unambiguous with respect to stationary boundary conditions satisfying $\lim_{x\to+\infty}\psi(x)=\delta$, where δ is proportional to the C^0 norm of $\delta F(\psi)$.] Any such a set of PDE's, however, would yield the same conclusions as the set satisfying $\delta F(\psi=0)=0$. Thus only perturbed PDE's unambiguous with respect to stationary boundary conditions satisfying $\lim_{x\to+\infty}\psi(x)=0$, and solutions satisfying the same will be considered.
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[37] This time must, of course, be small enough that nonlinear effects are still negligible. In order for there to exist a regime for which t is large enough that the asymptotic linear behavior is realized, but small enough that nonlinear effects remain negligible, the initial amplitude of the disturbance must be made sufficiently small.